Week 4

Recall: ① Matrix multiplication

$$(AB)_{ij} = \sum_{l=1}^{q} A_{il} B_{lj} \quad \text{for} \quad p \times q \text{ matrix } A, \quad q \times m \text{ matrix } B$$

② Row operations can be represented by Matrix multiplication.

Defn: A $n \times n$ matrix A is said to be invertible if $\exists$ $n \times n$ matrix B s.t. $AB = BA = I_n$

Call $B = A^{-1}$ in this case.

Eg: ① $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ then $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

② $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ then $B = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$.

"Because: $A \xrightarrow{-2R_2 + R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$

$\Rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = I_2$

checking: $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = I_2$ "

From Eg 2: I_2 and A are row equivalent ~~$\ast\ast$~~

Invertible $\overset{?}{\longleftrightarrow}$ solvability of $LS(A \mid b)$ (Next week!)

Transposes & symmetric matrix

defn: Given a $p \times q$ matrix $A = \begin{bmatrix} a_{11} & \cdots & a_{1q} \\ \vdots & & \\ a_{p1} & \cdots & a_{pq} \end{bmatrix}$,

the transpose of A is given by a $q \times p$ matrix A^t

where $(A^t)_{ij} = A_{ji} \quad \forall\, i = 1, \cdots, q;\ j = 1, \cdots, p$

$$A^t = \begin{bmatrix} a_{11} & \cdots & a_{p1} \\ \vdots & & \\ a_{1q} & \cdots & a_{pq} \end{bmatrix}$$

Eg: $A = \begin{bmatrix} 3 & 7 & 2 & -3 \\ -1 & 4 & 2 & 8 \\ 0 & 3 & -2 & 5 \end{bmatrix}$, $A^t = \begin{bmatrix} 3 & -1 & 0 \\ 7 & 4 & 3 \\ 2 & 2 & -2 \\ -3 & 8 & 5 \end{bmatrix}$

Defn: (special type of matrix)

A matrix A is symmetric if $A = A^t$. $\Rightarrow$ A must be a square matrix. $(p = q)$

$\left(\text{i.e., } A_{ij} = A_{ji} \quad \forall\, i = 1, 2, \cdots p;\ j = 1, 2, \cdots q = p.\right)$

Prop: Given $p \times q$ matrix A, B and $\lambda \in \mathbb{R}$,

1. $(A+B)^t = A^t + B^t$
2. $(\lambda A)^t = \lambda A^t$
3. $(A^t)^t = A$
4. $(AB)^t = B^t A^t$.

pf:

① $((A+B)^t)_{ij} = (A+B)_{ji} = A_{ji} + B_{ji}$

$= (A^t)_{ij} + (B^t)_{ij}$

② $((\lambda A)^t)_{ij} = (\lambda A)_{ji} = \lambda A_{ji} = \lambda (A^t)_{ij}$

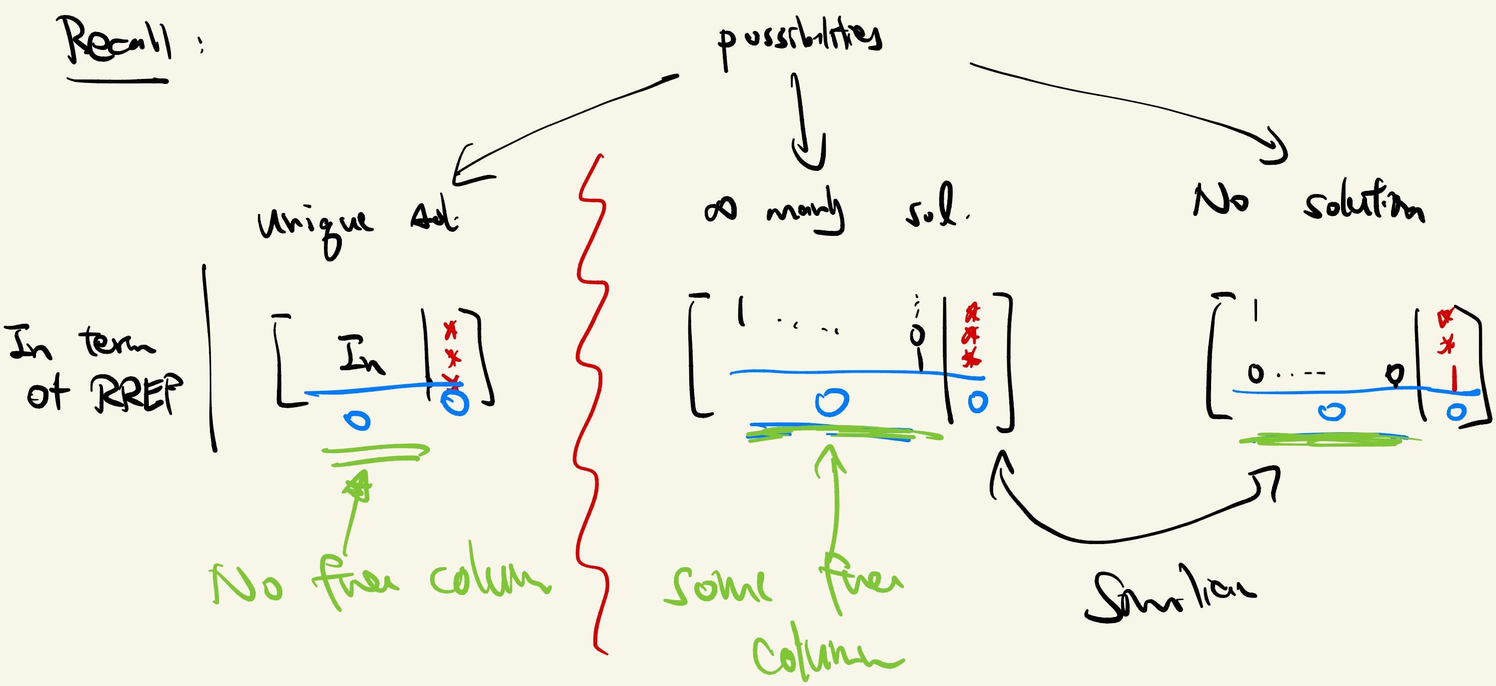
③ $((A^t)^t)_{ij} = (A^t)_{ji} = A_{ij}$

④ $((AB)^t)_{ij} = (AB)_{ji} = \sum_{l=1}^{p} A_{jl} B_{li}$

$= \sum_{l=1}^{p} (B^t)_{il} (A^t)_{lj} = (B^t A^t)_{ij}$

Back to system of linear eqn

Recall:



★ Important to consider coefficient matrix <u>only</u> (first)

Defn: A system of linear equation $LS(A,b)$ is homogeneous if $b=0$.

i.e.,
$$(S):\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = 0 \end{cases}$$

Defn: The homogeneous system corresponding to $LS(A,b)$ is defined to be $LS(A,0)$.

Thm: Any homogeneous system of linear equation are consistent.

pf: $x = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ is a trivial solution.

problem to be answered: Is $\vec{0}$ the unique solution??

$\boxed{\text{If } n=m}$

if unique, then

in RREF

$[A|0] \xrightarrow{\text{row}} [I_n|0]$

then

$[A|b]$ should admit solution for any b.

otherwise

$[A|0] \longrightarrow \left[\begin{array}{c|c} \ddots & \\ \hline 0 & 0 \end{array}\right]$

then

$[A|b]$ either admit ∞ many sol or No sol. depending on choice of b.

Thm: If a homogeneous system of linear equ. has m equation w/ n unknowns, then there are ∞ many sol if $m<n$.

~~☆☆~~ the non-trivial case : $n \le m$.

(trivial) Example when $n=m$ ☆☆.

① $\left[\begin{array}{cc|c} 1 & 0 & ? \\ 0 & 1 & ? \end{array}\right] = [A \mid b]$

then $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] = [A \mid 0]$ has $x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as the unique sol.

Then $[A|b]$ also has unique sol no matter what b is

② $\left[\begin{array}{cc|c} 1 & 1 & ? \\ 1 & 1 & ? \end{array}\right] = [A \mid b]$.

then $[A \mid 0] = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right]$ has $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ as RREF

w/ ∞ many solution.

$\therefore [A|b] = \left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 1 & 1 & b_2 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & 0 & b_2 - b_1 \end{array}\right]$

has ∞ sol if $b_1 = b_2$.

otherwise No sol. $\not\exists$

※※: Expectation from the example:

① If $n=m$ and $LS(A,0)$ admits only zero sol,

then $LS(A,b)$ admit solution for any b.

② if $n=m$ and $LS(A,0)$ admit non-zero sol.

then $LS(A,b)$ either has

(i) ∞ many sol

OR (ii) No solution

depending on what b is!!

** if $m>n$, there are no definite conclusion

eg: Ⓐ $[A|b] = \left[\begin{array}{cc|c} 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ \hline 0 & 1 & b_3 \end{array}\right]$ $m=3$, $n=2$.

① if $b_2=b_3$, it is redundant. $\Rightarrow$ unique

② if $b_2 \neq b_3$, it is impossible. $\Rightarrow$ No sol.

And $[A|0] = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right]$ admits only zero sol.

$\therefore$ No conclusion can be drawn by looking only at $LS(A, 0)$ if $m > n$.

Goal: Study whether $LS(A,0)$ has unique sol. or not.

Defn: The null space of A, denoted by $N(A)$, is the set of all vectors that are sol. to $LS(A,0)$.

That is $N(A) =$ sol. set of $\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = 0 \end{cases}$

Example:

$$[A|0] = \left[\begin{array}{cccc|c} 2 & 1 & 7 & -7 & 0 \\ -3 & 4 & -5 & -6 & 0 \\ 1 & 1 & 4 & -5 & 0 \end{array}\right] \xrightarrow{\text{Row op.}} \underbrace{\left[\begin{array}{cccc|c} 1 & 0 & 3 & -2 & 0 \\ 0 & 1 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]}_{\text{RREF}}$$

$\therefore$ solution set $= \left\{ \begin{bmatrix} -3s+2t \\ -s+3t \\ s \\ t \end{bmatrix} : s, t \in \mathbb{R} \right\}$

Then $N(A) = \left\{ \begin{bmatrix} -3s+2t \\ -s+3t \\ s \\ t \end{bmatrix} : s, t \in \mathbb{R} \right\}$ #

Example: $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 4 & 7 & 1 \end{bmatrix}$ $LS(A, 0)$: $m = 4$, $n = 3$

$$[A|0] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$ $\therefore$ sol. set $= \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

$\therefore N(A) = \{\vec{0}\}$

$m=n$: square matrix

<u>Defn</u>: If A = square matrix and $N(A) = \{0\}$, then

A is said to be non-singular.

otherwise A is singular.

<u>Example</u>: ①

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \longrightarrow \underset{\text{RREF}}{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}}$$

$\therefore N(A) = \left\{ \begin{bmatrix} -t \\ t \\ t \end{bmatrix} \middle| \, t \in \mathbb{R} \right\} \neq 0.$

$\therefore A$ is singular

②

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore N(A) = \{0\} \Rightarrow$ non-singular.

<u>Thm</u>: Suppose A is a square matrix and B is row equivalent to A s.t. B is RREF.

then A is non-singular if and only if $B = I_n$.

<u>pf:</u> $(\Rightarrow)$ if A = non-singular

then $[A \,|\, 0]$ has unique sol $\vec{0}$.

then $[B \,|\, 0]$ have no free column

then there are n pivot column $\Rightarrow B = I_n$

($\Leftarrow$) if $[B|0] = [I_n|0]$, then sol. $= \{0\}$

By thm before, sol of $[B|0]$ = sol of $[A|0]$ $= \{0\}$ #

Thm: If A is a square matrix, then A is non-singular iff $N(A) = \{0\}$

pf: A is non-singular iff $[A|0] \to [B|0] = [I_n|0]$

iff $N(A) = \{0\}$.

Thm: If A is a square matrix, then A is non-singular

Iff $LS(A, b)$ has unique sol for any b

pf: ($\Leftarrow$) Put $b=0 \Rightarrow N(A) = \{0\} \Rightarrow A$ is non-singular

($\Rightarrow$) If A is non-singular, then A and I_n are row eqv.

$\Rightarrow [A|b]$ and $[I_n|\tilde{b}]$ are row equivalent

and hence have same sol set

$\because [I_n|\tilde{b}]$ has unique sol, $[A|b]$ also uniq sol. #

Summary: A is square matrix $(n \times n)$, then the following are equivalent.

① A is non-singular

② A and I_n are row equivalent

③ $N(A) = 0$

④ $\forall b$, $LS(A, b)$ admits a unique sol.

Sol. set of $LS(A,b)$:

If x, y are both sol. to $LS(A,b)$,

then $\begin{cases} Ax = b \\ Ay = b \end{cases}$

$$\Rightarrow \quad A(x-y) = 0 \quad \Rightarrow \quad x-y \in N(A).$$

$\therefore$ ☆ If $N(A) = \{0\}$, then $x = y$

If $N(A) \neq \{0\}$, then $x = y + w$ for some $w \in N(A)$

Conversely, if $w \in N(A)$ and x is a sol. to $LS(A,b)$

then $A(w+x) = b \Rightarrow w+x$ is sol.

$\Downarrow$

Thm: Suppose w is a sol. to $LS(A,b)$ (i.e. $Aw = b$)

then y is also a sol. to $LS(A,b)$

iff $y = w + x$ for some $x \in N(A)$.

Example:

$$[A|b] = \left[\begin{array}{cccc|c} 2 & 1 & 7 & -7 & 8 \\ -3 & 4 & -5 & -6 & -12 \\ 1 & 1 & 4 & -5 & 4 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 4 & -5 & 4 \\ 0 & 1 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 3 & -2 & 4 \\ 0 & 1 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\therefore \text{sol set} = \left\{ \begin{bmatrix} 4+2t-3s \\ 3t-s \\ s \\ t \end{bmatrix} : t, s \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix} + s\begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix} : s, t \in \mathbb{R} \right\}$$

$$= \underline{\underline{\begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}}} + N(A).$$

$$N(A) = \left\{ t\begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix} + s\begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix} : s, t \in \mathbb{R} \right\}$$

(Not unique)

a choice of w

Eg: $[A \mid b] = \left[\begin{array}{ccc|c} 1 & 2 & 2 & 4 \\ 1 & 3 & 3 & 5 \\ 2 & 6 & 5 & 6 \end{array}\right]$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$$\therefore \text{sol} = \left\{ \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} + \vec{0} \right\} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} + \underbrace{N(A)}_{=\{0\}}$$