Pre-Lecture Notes of MATH-6042 at CUHK: Mathematical Theory and Numerical Simulation of Homogeneous Boltzmann Equation

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Abstract

In this course, the development of the spatially homogeneous theory to the Boltzmann equation will be briefly introduced, especially for the well-posedness result of the Cauchy problem in the space of probability measure. On the other side, the numerical simulation about the homogeneous Boltzmann equation, mainly the deterministic Spectral Method will also be presented; furthermore, some corresponding stability/error analysis frameworks will be discussed in a suitable manner.

1 Personal Statement

The lecture note is based on the MATH-6042 course delivered by the author in the Term 2, 2021-2022 at CUHK. The main prerequisites are a reasonable acquaintance with functional analysis, i.e., elementary topology, Fourier transform, and so forth. Preliminary knowledge about the Boltzmann equation is literally preferred, though the brief introduction will be provided at the beginning.

Due to the current limitation of the author, most likely, there are still at places inadequacies, inconsistency of notations, inadvertently omitted references... Therefore, the lecture note will be constantly updated and frequently uploaded on the website of the author, and hopefully continue to cover up the most recent results of this topic with time evolution.

Any correction and comment will be very welcomed from the readers for further improvement of the lecture note.

Contents

2 Introduction of Boltzmann Equation

2.1 The Spatially Homogeneous Boltzmann Equation

In the spatially homogeneous theory of the Boltzmann equation, one is interested in the solution $f(t,x,v)$ which does not depend on the x space variable. This view of point is pretty common in physics, especially when it comes to the problems focusing on the collision operator, as the collision integral operator only acts on the velocity dependence. On the other hand, the interests towards the spatially homogeneous study also arise from the numerical analysis, since almost all numerical schemes succeed from the splitting of the transport step and collision step.

In this case, the homogeneous Boltzmann equation in $\mathbb{R}^3$ reads:

$$\partial_t f(t,v) = Q(f,f)(t,v), \tag{2.1}$$

with the non-negative initial condition,

$$f(0,v) = F_0(v), \tag{2.2}$$

where the unknown $f = f(t,v)$ is regarded as the density function of a probability distribution, or more generally, a probability measure; and the initial datum F_0 is also assumed to be a non-negative probability measure on $\mathbb{R}^3$.

The right hand side of (2.1) is the so-called Boltzmann collision operator,

$$\begin{aligned} Q(f,f)(v) &= \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B_\sigma(v-v_*,\sigma)\,[f(v')f(v'_*) - f(v)f(v_*)]\,\mathrm{d}\sigma\,\mathrm{d}v_* \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B_\omega(v-v_*,\sigma)\,[f(v')f(v'_*) - f(v)f(v_*)]\,\mathrm{d}\omega\,\mathrm{d}v_*, \end{aligned} \tag{2.3}$$

where (v', v'_*) and (v, v_*) represent the velocity pairs before and after a collision, which satisfy the conservation of momentum and energy:

$$v' + v'_* = v + v_*, \quad |v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2, \tag{2.4}$$

so that (v', v'_*) can be expressed in terms of (v, v_*) as

$$\begin{aligned} v' &= \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \quad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma, \\ \text{or} \quad v' &= v - v[(v-v_*\cdot\omega)]\omega, \quad v'_* = v + v[(v-v_*\cdot\omega)]\omega, \end{aligned} \tag{2.5}$$

where both of σ and ω are a vector varying over the unit sphere $\mathbb{S}^2$. And this also easily implies the relations

$$v\cdot v_* = v'\cdot v'_*, \quad |v-v_*| = |v'-v'_*|, \quad (v-v_*)\cdot\omega = -(v'-v'_*)\cdot\omega. \tag{2.6}$$

and

$$|\langle v-v_*,\omega\rangle| = |v-v_*|\cos\alpha = |v-v_*|\cos\left(\frac{\pi-\theta}{2}\right) = |v-v_*|\sin\frac{\theta}{2}, \tag{2.7}$$

where α denotes that angle between $v-v_*$ and ω.

Next, we have a more general relation between the σ- and ω- representation in the sense that,

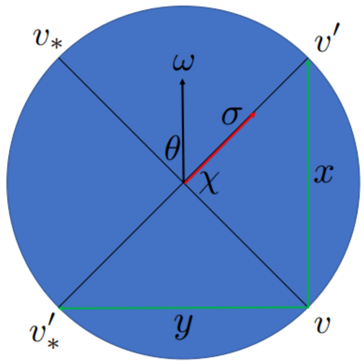

Figure 1: Velocity and unit vector during a classical elastic collision.

Lemma 2.1. *For the change of variables:*

$$\sigma = \frac{v - v_*}{|v - v_*|} - 2\left\langle \frac{v - v_*}{|v - v_*|}, \omega \right\rangle, \tag{2.8}$$

it has the Jacobian

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\omega} = 2^{d-1}\left|\left\langle \frac{v - v_*}{|v - v_*|}, \omega \right\rangle\right|^{d-2}. \tag{2.9}$$

Proof. Fix the unitary vector $\hat{q} = \frac{v-v_*}{|v-v_*|}$ and $\left\langle \frac{v-v_*}{|v-v_*|}, \omega \right\rangle = \hat{q} \cdot \omega$, then the change of variables can be regarded as a map $\sigma(\omega) : \mathbb{S}^{d-1} \longmapsto \mathbb{S}^{d-1}$ give by

$$\sigma(\omega) = \hat{q} - 2\left(\hat{q} \cdot \omega\right)\omega. \tag{2.10}$$

Let $\mathcal{O}_{\hat{q}}$ be the orthogonal space to $\hat{q}$, α be the angle between $\hat{q}$ and ω, and θ be the angle between $\hat{q}$ and σ. In this way, one may write

$$\omega = \cos\alpha\hat{q} + \omega_o, \quad \sigma = \cos\theta\hat{q} + \sigma_o \tag{2.11}$$

where $\omega_o, \sigma_o \in \mathcal{O}_{\hat{q}}$. Using the spherical coordinates with north pole given by $\hat{q}$, the measures $\mathrm{d}\omega$ and $\mathrm{d}\sigma$ are given by

$$\mathrm{d}\omega = \sin^{d-2}\alpha \mathrm{d}\hat{\omega}_o \mathrm{d}\alpha, \quad \mathrm{d}\sigma = \sin^{d-2}\theta \mathrm{d}\hat{\sigma}_o \mathrm{d}\theta \tag{2.12}$$

where the measures $\mathrm{d}\hat{\omega}_o$ and $\mathrm{d}\hat{\sigma}_o$ are the Lebesgue measure in $\mathbb{S}^{d-2}(\hat{q})$ parameterized with the vectors ω_o, σ_o respectively. Directly from the expression of the map, we find,

$$\cos\theta = \hat{q} \cdot \sigma = 1 - 2\left(\hat{q} \cdot \omega\right)^2 = 1 - 2\cos^2\alpha. \tag{2.13}$$

Then, it follows by direct differentiation that

$$-\sin\theta\mathrm{d}\theta = 4\cos\alpha\sin\alpha\mathrm{d}\alpha. \tag{2.14}$$

Now, choose a orthonormal base $\{\xi_j\}_{j=1}^{d-2}$ for $\mathcal{O}_{\hat{q}}$. Compute again using the explicit expression of the map

$$\begin{aligned} \sigma_o = & \sum_{j=1}^{d-2}\left(\sigma \cdot \xi_j\right)\xi_j = -2\left(\hat{q} \cdot \omega\right)\sum_{j=1}^{d-2}\left(\omega \cdot \xi_j\right)\xi_j \\ = & -2\left(\hat{q} \cdot \omega\right)\omega_o = -2\cos\alpha\omega_o. \end{aligned} \tag{2.15}$$

Thus, $\hat{\omega_o} = \hat{\sigma_o}$, and as a consequence, $\mathrm{d}\hat{\omega_o} = \mathrm{d}\hat{\sigma_o}$. Gathering these relations all together and using the basic trigomometry

$$\mathrm{d}\omega = \left(\frac{\sin\alpha}{\sin\theta}\right)^{d-3} \frac{\mathrm{d}\sigma}{4|\cos\alpha|} = \frac{\mathrm{d}\sigma}{2^{d-1}|\cos\alpha|^{d-2}}. \tag{2.16}$$

This completes the proof. □

Lemma 2.2. *Fix $\sigma \in \mathbb{S}^{d-1}$ and $q = v - v_*$, the map $u : \mathbb{R}^d \longmapsto \mathbb{R}^d$ given by*

$$u(q) = \frac{q + |q|\sigma}{2} \tag{2.17}$$

has Jacobian

$$\frac{\mathrm{d}u}{\mathrm{d}q} = \frac{1 + \sigma \cdot \hat{q}}{2^d}. \tag{2.18}$$

Proof. Choose an orthonormal base $\{\sigma, \xi_j\}$ with $2 \le j \le d$. Then, the coordinates of this change of variables are

$$\begin{aligned} z_1 =& z \cdot \sigma = \frac{1}{2}\left(q \cdot \sigma + |q|\right) = \frac{1}{2}\left(q_1 + |q|\right), \\ z_j =& z \cdot \xi_j = \frac{1}{2} q_j, \quad j = 2, ..., d. \end{aligned} \tag{2.19}$$

Thus,

$$\frac{\partial z_1}{\partial q_1} = \frac{1}{2}\left(1 + \hat{q} \cdot \sigma\right), \quad \frac{\partial z_j}{\partial q_l} = \frac{1}{2}\delta_{jl}, \quad j = 2, ..., d. \tag{2.20}$$

and, therefore,

$$\frac{\mathrm{d}z}{\mathrm{d}q} = \prod_{j=1}^{d} \left|\frac{\partial z_j}{\partial q_j}\right| = \frac{1 + \hat{q} \cdot \sigma}{2^d}. \tag{2.21}$$

□

2.2 The Boltzmann collision kernel.

The collision kernel B is a non-negative function that depends only on $|v - v_*|$ and cosine of the deviation angle θ, whose specific form can be determined from the intermolecular potential using classical scattering theory. For example, in the case of **Inverse Power Law Potentials** $U(r) = r^{-(\ell-1)}, 2 < \ell < \infty$, where r is the distance between two interacting particles, B can be separated as the kinetic part and angular part:

$$B(v - v_*, \sigma) = B(|v - v_*|, \cos\theta) = b(\cos\theta)\Phi(|v - v_*|), \quad \cos\theta = \frac{\sigma \cdot (v - v_*)}{|v - v_*|}, \tag{2.22}$$

where the kinetic part

$$\Phi(|v - v_*|) = |v - v_*|^\gamma = \begin{cases} \gamma > 0, & \text{Hard potential,} \\ \gamma = 0, & \text{Maxwellian gas,} \\ \gamma < 0, & \text{Soft potential.} \end{cases} \quad \gamma = \frac{\ell - 5}{\ell - 1} > -3 \text{ (when } d = 3),$$

and the angular part

$$\sin^{d-2}\theta b(\cos\theta)\big|_{\theta\to 0^+} \sim K\theta^{-1-\nu}, \quad 0<\nu=\frac{2}{\ell-1}<2 \text{ (when } d=3). \tag{2.23}$$

The kernel (2.22) encompasses a wide range of potentials, among which we mention three extreme cases [8]:

(i) $\ell=\infty$, $\gamma=1$, $\nu=0$ corresponds to the hard spheres, where B is only proportional to $|v-v_*|$,

$$B(|v-v_*|,\cos\theta)=K|v-v_*|, \quad K>0; \tag{2.24}$$

(ii) $\ell=2$, $\gamma=-3$, $\nu=2$ corresponds to the Coulomb interaction, where B is given by the famous Rutherford formula,

$$B(|v-v_*|,\cos\theta)=\frac{1}{|v-v_*|^3\sin^4(\theta/2)}; \tag{2.25}$$

(iii) $\ell=5$, $\gamma=0$, $\nu=\frac{1}{2}$ corresponds to the literally physical Maxwellian gas, where B does not depend on relative velocity $|v-v_*|$,

$$B(|v-v_*|,\cos\theta)=b\left(\frac{v-v_*}{|v-v_*|}\cdot\sigma\right)=b(\cos\theta). \tag{2.26}$$

However, instead of this very special case above, we are interested in the more general case $B=b(\cos\theta)$, not depending on $|v-v_*|$ that,

$$\gamma=0, \quad 0<\nu<2, \tag{2.27}$$

which is called Maxwellian molecules type.

The range of deviation angle θ, namely the angle between pre- and post-collisional velocities, is a full interval $[0,\pi]$, but it is customary to restrict it to $[0,\pi/2]$ mathematically, replacing $b(\cos\theta)$ by its "symmetrized" version [21]:

$$\left[b(\cos\theta)+b(\cos(\pi-\theta))\right]\mathbf{1}_{0\le\theta\le\frac{\pi}{2}}, \tag{2.28}$$

which amounts more or less to forbidding the exchange of particles.

Another physically interesting example that is not explicit at all has been called **Debye-Yukawa Potential** $U(r)=\mathrm{e}^{-r}/r$, also asymptotically behaving as $\theta\to 0$:

$$\sin^{d-2}\theta B(|v-v_*|,\cos\theta)\Big|_{\theta\to 0^+} \sim K|v-v_*|\theta^{-1}|\log\theta^{-1}|. \tag{2.29}$$

2.3 Cutoff VS Non-cutoff

As it has been long known, the main difficulty in establishing the well-posedness result for Boltzmann equation is that the singularity of the collision kernel b is not locally integrable in $\sigma\in\mathbb{S}^2$. To avoid this, H. Grad gave the integrable assumption on the collision kernel b_c by a "**Cutoff** " near singularity:

$$\int_{\mathbb{S}^2} b_c\left(\frac{v-v_*}{|v-v_*|}\cdot\sigma\right)\mathrm{d}\sigma=2\pi\int_0^{\frac{\pi}{2}} b_c(\cos\theta)\sin\theta\,\mathrm{d}\theta<\infty. \tag{2.30}$$

However, the full singularity condition for the collision kernel with **Non-cutoff Assumption** is implicitly defined for the angular collision part $b(\cos\theta)$, which asymptotically behaves as $\theta \to 0^+$,

$$\sin\theta b(\cos\theta)\big|_{\theta\to 0^+} \sim K\theta^{-1-\nu}, \quad \nu = \frac{2}{\ell - 1}, \quad 0 < \nu < 2 \quad \text{and} \quad K > 0, \tag{2.31}$$

or in "symmetrized" manner,

$$\exists \alpha_0 \in (0, 2], \quad \text{such that} \quad \int_0^{\frac{\pi}{2}} \sin^{\alpha_0}\left(\frac{\theta}{2}\right) b(\cos\theta)\sin\theta \mathrm{d}\theta < \infty, \tag{2.32}$$

which can handle the strongly singular kernel b in (2.31) with some $0 < \nu < 2$ and $\alpha_0 \in (\nu, 2]$. Besides, we further illustrate that the non-cutoff assumption (2.32) can be rewritten as

$$(1-s)^{\frac{\alpha_0}{2}} b(s) \in L^1[0, 1), \quad \text{for } \alpha_0 \in (0, 2], \tag{2.33}$$

by means of the transformation of variable $s = \cos\theta$ in the symmetric version of b. As mentioned in [15, Remark 1], the full non-cutoff assumption (2.32), or equivalently (2.33), is the extension of the mild non-cutoff assumption of the collision kernel b used in [14], namely,

$$(1-s)^{\frac{\alpha_0}{4}} (1+s)^{\frac{\alpha_0}{4}} b(s) \in L^1(-1, 1), \quad \text{for } \alpha_0 \in (0, 2]. \tag{2.34}$$

2.4 The Weak Formulation and Conservation Law

To derive the weak formulation, a universal tool (so-called *Pre-postcollisional change of variables*) is frequently used, which is an involutive change of variables with unit Jacobian,

$$(v, v_*, \sigma) \to (v', v'_*, \hat{q}), \tag{2.35}$$

where $\hat{q}$ is the unit vector along the relative velocity $q := v - v_*$,

$$\hat{q} = \frac{v - v_*}{|v - v_*|}. \tag{2.36}$$

On the other hand, since $\sigma = (v' - v'_*)/|v' - v'_*|$, the change of variables (2.35) formally amounts to the change of (v, v_*) and (v', v'_*). Hence, under suitable integrability conditions on the measurable function F,

$$\begin{aligned}
&\int_{\mathbb{S}^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} B(|v - v_*|, \hat{q}\cdot\sigma) F(v, v_*, v', v'_*)\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\sigma \\
=&\int_{\mathbb{S}^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} B(|v - v_*|, \hat{q}\cdot\sigma) F(v, v_*, v', v'_*)\,\mathrm{d}v'\,\mathrm{d}v'_*\,\mathrm{d}\sigma \\
=&\int_{\mathbb{S}^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} B(|v - v_*|, \hat{q}\cdot\sigma) F(v', v'_*, v, v_*)\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\sigma,
\end{aligned} \tag{2.37}$$

where the fact $|v' - v'_*| = |v - v_*|, \sigma\cdot\hat{q} = \hat{q}\cdot\sigma$ is used to keep the arguments of collision kernel $B(v - v_*, \sigma) = B(|v - v_*|, \hat{q}\cdot\sigma)$ unchanged. Note that the change of variables $(v, v_*) \to (v', v'_*)$ works for a fixed ω but is illegal for any given σ.

With the help of this microreversiblity of velocity from (v, v) to (v', v'_*), which leaves the collision kernel B invariant, we can obtain the following weak form for the Boltzmann collision operator.

Proposition 2.3. *For any test function ϕ that is an arbitrarily continuous function of the velocity v,*

$$\begin{aligned}\int_{\mathbb{R}^3} Q(f,f)\phi \, \mathrm{d}v &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-v_*,\sigma)(f'f'_* - ff_*)\phi \, \mathrm{d}\sigma \, \mathrm{d}v_* \, \mathrm{d}v \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-v_*,\sigma)ff_*(\phi'-\phi) \, \mathrm{d}\sigma \, \mathrm{d}v_* \, \mathrm{d}v \\ &= \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-v_*,\sigma)ff_*(\phi'+\phi'_*-\phi-\phi_*) \, \mathrm{d}\sigma \, \mathrm{d}v_* \, \mathrm{d}v \\ &= \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-v_*,\sigma)(f'f'_*-ff_*)(\phi+\phi_*-\phi'-\phi'_*) \, \mathrm{d}\sigma \, \mathrm{d}v_* \, \mathrm{d}v. \end{aligned} \tag{2.38}$$

2.5 Boltzmann's H–Theorem and Equilibrium State

Recall the weak formulation (2.38) of the Boltzmann equation as in (2.3), there is an immediate consequence for a solution f to the Boltzmann equation that, whenever ϕ satisfies the functional equation,

$$\forall (v, v_*, \sigma) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2, \qquad \phi(v') + \phi(v'_*) = \phi(v) + \phi(v_*), \tag{2.39}$$

then, we at least formally have,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} f(t,v)\phi(v) \, \mathrm{d}v = \int_{\mathbb{R}^3} Q(f,f)\phi \, \mathrm{d}v = 0, \tag{2.40}$$

and this kind of ϕ is usually called the *collision invariant*.

Since the mass, momentum and energy are conserved during the classical elastic collisions, it is natural to find that the functions $1, v_j, 1 \le j \le 3$, and $|v|^2$ and any linear combination of them are the collision invariants, which can be actually shown as the only collision invariants. Together with the weak form, this leads to the formal conservation law of the Boltzmann equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} f(t,v) \begin{pmatrix} 0 \\ v_j \\ |v|^2 \end{pmatrix} \mathrm{d}v = \int_{\mathbb{R}^3} Q(f,f)(t,v) \begin{pmatrix} 0 \\ v_j \\ |v|^2 \end{pmatrix} \mathrm{d}v = 0, \qquad 1 \le j \le 3. \tag{2.41}$$

In particular, at a given time t, one can define the local density ρ, the local macroscopic velocity u, and the local temperature T, by

$$\rho = \int_{\mathbb{R}^3} f(t,v) \, \mathrm{d}v, \quad \rho u = \int_{\mathbb{R}^3} f(t,v) v \, \mathrm{d}v, \quad \rho|u|^2 + d\rho T = \int_{\mathbb{R}^3} f(t,v)|v|^2 \, \mathrm{d}v, \tag{2.42}$$

then the equilibrium is the *Maxwellian* distribution,

$$\mathcal{M}(v) = \mathcal{M}^f(v) = \frac{1}{(2\pi T)^{3/2}} \mathrm{e}^{-\frac{|v-u|^2}{2T}}. \tag{2.43}$$

If not caring about the integrability issues, we select the test function $\phi = \log f$ into the weak form (2.38), and consider the properties of the logarithm function, to find that

$$\begin{aligned} -\int_{\mathbb{R}^3} Q(f,f)\ln f \,\mathrm{d}v =& D(f) \\ =& \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-v_*,\sigma)(f'f'_* - ff_*)\ln\frac{f'f'_*}{ff_*} \geq 0 \end{aligned} \tag{2.44}$$

due to the fact that the function $(X,Y) \longmapsto (X-Y)(\ln X - \ln Y)$ is always non-negative. Thus, if we introduce Boltzmann's H–functional,

$$H(f) = \int_{\mathbb{R}^3} f \ln f \,\mathrm{d}v, \tag{2.45}$$

then the $H(f)$ will evolve in time because of the collisional effect that

$$\frac{\mathrm{d}}{\mathrm{d}t} H(f(t,\cdot)) = -D(f(t,\cdot)) \leq 0, \tag{2.46}$$

which is the well-known Boltzmann's H–Theorem: the H–functional, or entropy, is non-increasing with time evolution.

And the equality holds if and only if $\ln f$ is a collision invariant, i.e., $f = \exp(a + bv + c|v|^2)$ with a, b, c being all constants.

2.6 Fourier Transform of the Collision Operator (Bobylev Identity)

The Fourier transformation has been widely used in the analysis of various kind of partial differential equations. However, it used to be very painful to find an elegant representation of the Boltzmann equation in the Fourier space, even though the Boltzmann operator possesses a nice weak formulation. Thanks to A. V. Bobylev, this problem turned out not as intricate as one may imagine, at least for the Maxwellian molecules. Since then, the so-called "Bobylev Identity" has become an extremely powerful technique in the study of the Boltzmann equation, especially in the case of spatially homogeneous theory.

Proposition 2.4. *Consider the Boltzmann collision operator $Q(g,f)$ with its collision kernel B being the Maxwellian molecule b, i.e., B does not depend on $|v - v_*|$,*

$$Q(f,f)(v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b\left(\frac{v-v_*}{|v-v_*|}\cdot\sigma\right)[f(v')f(v'_*) - f(v)f(v_*)]\,\mathrm{d}\sigma\,\mathrm{d}v_*. \tag{2.47}$$

Then, the following formulas hold,

$$\begin{aligned} \mathcal{F}\left[Q^+(g,f)\right](\xi) &= \int_{\mathbb{S}^2} b\left(\frac{\xi}{|\xi|\cdot\sigma}\right)\hat{g}(\xi^-)\hat{f}(\xi^+)\,\mathrm{d}\sigma, \\ \mathcal{F}\left[Q^-(g,f)\right](\xi) &= \int_{\mathbb{S}^2} b\left(\frac{\xi}{|\xi|\cdot\sigma}\right)\hat{g}(0)\hat{f}(\xi)\,\mathrm{d}\sigma, \end{aligned} \tag{2.48}$$

where,

$$\xi^+ = \frac{\xi}{2} + \frac{|\xi|}{2}\sigma, \qquad \xi^- = \frac{\xi}{2} - \frac{|\xi|}{2}\sigma. \tag{2.49}$$

Proof. By performing the weak formulation, for any test function ϕ, we have,

$$\int_{\mathbb{R}^3} Q^+(g,f)(v)\phi(v)\mathrm{d}v = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b\left(\frac{v-v_*}{|v-v_*|}\cdot\sigma\right) g(v_*)f(v)\phi(v')\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v. \quad (2.50)$$

Selecting $\phi(v) = \mathrm{e}^{-\,\mathrm{i}v\cdot\xi}$ in the identity above, we have

$$\begin{aligned}&\mathcal{F}\left[Q^+(g,f)\right](\xi)\\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b\left(\frac{v-v_*}{|v-v_*|}\cdot\sigma\right) g(v_*)f(v)\,\mathrm{e}^{-\,\mathrm{i}\left(\frac{v+v_*}{2}+\frac{|v-v_*|}{2}\sigma\right)\cdot\xi}\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b\left(\frac{v-v_*}{|v-v_*|}\cdot\sigma\right) g(v_*)f(v)\,\mathrm{e}^{-\,\mathrm{i}\frac{v+v_*}{2}\cdot\xi}\,\mathrm{e}^{-\,\mathrm{i}\frac{|v-v_*|}{2}\sigma\cdot\xi}\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v,\end{aligned} \quad (2.51)$$

according to the general change of variable,

$$\int_{\mathbb{S}^2} F(k\cdot\sigma, l\cdot\sigma)\,\mathrm{d}\sigma = \int_{\mathbb{S}^2} F(l\cdot\sigma, k\cdot\sigma)\,\mathrm{d}\sigma, \quad |l| = |k| = 1, \quad (2.52)$$

due to the existence of an isometry on $\mathbb{S}^2$ exchanging l and k, we have, by exchanging the rule of $\frac{\xi}{|\xi|}$ and $\frac{v-v_*}{|v-v_*|}$,

$$\begin{aligned}&\int_{\mathbb{S}^2} g(v_*)f(v)b\left(\frac{v-v_*}{|v-v_*|}\cdot\sigma\right)\mathrm{e}^{-\,\mathrm{i}\frac{|v-v_*|}{2}\sigma\cdot\xi}\,\mathrm{d}\sigma\\ &= \int_{\mathbb{S}^2} g(v_*)f(v)b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)\mathrm{e}^{-\,\mathrm{i}\frac{|\xi|}{2}\sigma\cdot(v-v_*)}\,\mathrm{d}\sigma\end{aligned} \quad (2.53)$$

Thus,

$$\begin{aligned}&\mathcal{F}\left[Q^+(g,f)\right](\xi)\\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} g(v_*)f(v)b\left(\frac{v-v_*}{|v-v_*|}\cdot\sigma\right)\mathrm{e}^{-\,\mathrm{i}\frac{v+v_*}{2}\cdot\xi}\,\mathrm{e}^{-\,\mathrm{i}\frac{|v-v_*|}{2}\sigma\cdot\xi}\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} g(v_*)f(v)b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)\mathrm{e}^{-\,\mathrm{i}\frac{v+v_*}{2}\cdot\xi}\,\mathrm{e}^{-\,\mathrm{i}\frac{|\xi|}{2}\sigma\cdot(v-v_*)}\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} g(v_*)f(v)b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)\mathrm{e}^{-\,\mathrm{i}v\cdot\left(\frac{\xi}{2}+\frac{|\xi|}{2}\sigma\right)}\,\mathrm{e}^{-\,\mathrm{i}v_*\cdot\left(\frac{\xi}{2}-\frac{|\xi|}{2}\sigma\right)}\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\\ &= \int_{\mathbb{S}^2} b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)\hat{f}(\xi^+)\hat{g}(\xi^-)\,\mathrm{d}\sigma,\end{aligned} \quad (2.54)$$

where, unlike the elastic case, the ξ^+ and ξ^- are defined as

$$\xi^+ = \frac{\xi}{2} + \frac{|\xi|}{2}\sigma, \qquad \xi^- = \frac{\xi}{2} - \frac{|\xi|}{2}\sigma. \quad (2.55)$$

And the formula for $\mathcal{F}\left[Q^-(g,f)\right](\xi)$ is then easily obtained by the same kind of but more simpler computations. □

For a given probability measure F or its density function f, we define the corresponding characteristic function $\varphi(\xi)$ by the Fourier transform:

$$\varphi(\xi) = \hat{f}(\xi) := \int_{\mathbb{R}^3} \mathrm{e}^{-\,\mathrm{i}v\cdot\xi} f(v)\,\mathrm{d}v = \int_{\mathbb{R}^3} \mathrm{e}^{-\,\mathrm{i}v\cdot\xi}\,\mathrm{d}F(v), \quad (2.56)$$

where the f is regarded as the distribution density function of the cumulative distribution function F in the sense of Radon-Nikodym derivative.

And its inversion formula by normalization writes

$$f(v) = \int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}v\cdot\xi} \hat{f}(\xi)\, \mathrm{d}\xi = \int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}v\cdot\xi} \varphi(\xi)\, \mathrm{d}\xi. \tag{2.57}$$

3 Corresponding and Relevant Materials

The following materials are, in chronological order, referred to the development of the study about the solution to spatially homogeneous Boltzmann equation as a probability measure, where the Fourier Transformation plays a critical role.

KQ: This list is not intended to be completely covered in the mini-course, which is definitely impossible, but to partly reflect the history and hopefully present a big picture about how the research of the homogeneous Boltzmann equation in probability measure sense developed: from cutoff to non-cutoff, from the Maxwellian molecule to hard/soft potential, from higher-order moments requirement to lower-order... The selection is biased in favor of personal taste.

[1] A. V. Bobylev. A class of invariant solutions of the Boltzmann equation. *Dokl. Akad. Nauk SSSR*, 231(3):571–574, 1976.

[2] A. V. Bobylev. The theory of the nonlinear spatially uniform Boltzmann equation for Maxwell molecules. *Soviet Sci. Rev. Sect. C: Math. Phys. Rev., 7*, 111–233, 1988.

[3] G. Gabetta, G. Toscani, and B. Wennberg. Metrics for probability distributions and the trend to equilibrium for solutions of the Boltzmann equation. *J. Stat. Phys.*, 81(5-6):901–934, 1995.

[4] A. Pulvirenti and G. Toscani. The theory of the nonlinear Boltzmann equation for Maxwell molecules in Fourier representation. *Ann. Mat. Pura Appl. (4)*, 171:181-204, 1996.

[5] E. A. Carlen, E. Gabetta, and G. Toscani. Propagation of smoothness and the rate of exponential convergence to equilibrium for a spatially homogeneous Maxwellian gas. *Comm. Math. Phys.*, 199(3):521–546, 1999.

[6] G. Toscani and C. Villani. Probability metrics and uniqueness of the solution to the Boltzmann equation for a Maxwell gas. *J. Stat. Phys.*, 94(3-4):619–637, 1999.

[7] A. V. Bobylev and C. Cercignani. Self-similar solutions of the Boltzmann equation and their applications. *J. Stat. Phys.*, 106(5-6):1039–1071, 2002.

[8] C. Villani. A review of mathematical topics in collisional kinetic theory. In S. Friedlander and D. Serre, editors, *Handbook of Mathematical Fluid Mechanics*, volume I, pages 71–305. North-Holland, 2002.

[9] L. Desvillettes. About the use of the Fourier transform for the Boltzmann equation. volume 2*, pages 1–99. 2003. Summer School on "Methods and Models of Kinetic Theory" (M&MKT 2002).

[10] C. Villani. Mathematics of granular materials. *J. Stat. Phys.*, 124(2-4):781–822, 2006.

[11] J. A. Carrillo, G. Toscani Contractive probability metrics and asymptotic behavior of dissipative kinetic equations. *Riv. Mat. Univ. Parma*, 7(6):75–198, 2007.

[12] A. V. Bobylev, C. Cercignani, and I. M. Gamba. On the self-similar asymptotics for generalized nonlinear kinetic Maxwell models. *Comm. Math. Phys.*, 291(3):599–644, 2009.

[13] N. Fournier and C. Mouhot. On the well-posedness of the spatially homogeneous Boltzmann equation with a moderate angular singularity. *Comm. Math. Phys.*, 289(3):803–824, 2009.

[14] M. Cannone and G. Karch. Infinite energy solutions to the homogeneous Boltzmann equation. *Comm. Pure Appl. Math.*, 63(6):747–778, 2010.

[15] Y. Morimoto. A remark on Cannone-Karch solutions to the homogeneous Boltzmann equation for Maxwellian molecules. *Kinet. Relat. Models*, 5(3):551–561, 2012.

[16] X. Lu and C. Mouhot. On measure solutions of the Boltzmann equation, part I: moment production and stability estimates. *J. Differential Equations*, 252(4):3305–3363, 2012.

[17] M. Cannone and G. Karch. On self-similar solutions to the homogeneous Boltzmann equation. *Kinet. Relat. Models*, 6(4):801–808, 2013.

[18] X. Lu and C. Mouhot. On measure solutions of the Boltzmann equation, Part II: Rate of convergence to equilibrium. *J. Differential Equations*, 258(11):3742–3810, 2015.

[19] N. Fournier. Finiteness of entropy for the homogeneous Boltzmann equation with measure initial condition. *Ann. Appl. Probab.*, 25(2):860–897, 2015.

[20] Y. Morimoto, S. Wang, and T. Yang. A new characterization and global regularity of infinite energy solutions to the homogeneous Boltzmann equation. *J. Math. Pures Appl. (9)*, 103(3):809–829, 2015.

[21] Y. Morimoto, S. Wang, and T. Yang. Measure valued solutions to the spatially homogeneous Boltzmann equation without angular cutoff. *J. Stat. Phys.*, 165(5):866–906, 2016.

[22] Y. Morimoto, T. Yang, and H. Zhao. Convergence to self-similar solutions for the homogeneous Boltzmann equation. *J. Eur. Math. Soc.*, 19(8):2241–2267, 2017.

[23] K. Qi. Measure Valued Solution to the Spatially Homogeneous Boltzmann Equation with Inelastic Long-Range Interactions. *Preprint*, 2020.

[24] A. V. Bobylev, A. Nota, J. J. L. Velázquez. Self-similar asymptotics for a modified Maxwell-Boltzmann equation in systems subject to deformations. *Comm. Math. Phys.*, 380(1):409–448, 2020.

[25] K. Qi. On the measure valued solution to the inelastic Boltzmann equation with soft potentials. *J. Stat. Phys.*, 183(27), 2021.