Lecture_3

Thursday, 17 February 2022 2:32 pm

Weak solution Theory:
$$\begin{cases} \frac{\partial f(t,v)}{\partial t} = Q(f,f)(t,v) \\ f(t=0,v) = f_0(v) \end{cases} \quad (*)$$

Definition: (Weak solution $-2<\gamma<1$) $\qquad B(|v-v_*|,w) = b(\cos\theta)|v-v_*|^{\gamma}$

Let f_0 be a function defined on $\mathbb{R}^3$, with finite mass, energy, and entropy. We say that $(t,v) \longmapsto f(t,v)$ is weak solution of the Cauchy Problem $(*)$, if the following conditions are fulfilled.

$\boxed{-2<\gamma<1}$

$-3<\gamma<-2$ very soft

(a) $f \geq 0$. $f \in C(\mathbb{R}^+, (C_c^\infty(\mathbb{R}^3))^*)$ $\qquad \mathcal{D}'$

$\forall t \geq 0$, $f(t,v) \in L_2^1 \cap L\log L$

$f \in L^1([0,T], L^1_{2+\gamma})$

$$\begin{cases} L_2^1(\mathbb{R}^3) := \{f \mid \int |f(v)|(1+|v|^2)dv < \infty\} \\ L^1_{2+\gamma}(\mathbb{R}^3) := \{f \mid \int |f(v)|(1+|v|^{2+\gamma})dv < \infty\} \end{cases}$$

(b) $f_0(v) = f(t=0,v)$ $\qquad L\log L = \{f \mid \int_{\mathbb{R}^3} f\log f\, dv < \infty\}$

(c) $\forall t \geq 0$, $\int_{\mathbb{R}^3} f(t,v)\phi\, dv = \int_{\mathbb{R}^3} f_0(v)\phi\, dv$, for $\phi = 1, v_i, |v|^2$. $\quad 1 \leq i \leq 3$

$$\int_{\mathbb{R}^3} f\log f\, dv \leq \int_{\mathbb{R}^3} f_0 \log f_0\, dv.$$

(d) $\forall \phi(t,v) \in C^1(\mathbb{R}^+, C_c^\infty(\mathbb{R}^3))$, $\forall t \geq 0$

$$\int_{\mathbb{R}^3} f(t,v)\phi(t,v)dv - \int_{\mathbb{R}^3} f_0(v)\phi_0(v)dv - \int_0^t \int_{\mathbb{R}^3} f(\tau)\partial_t\phi(\tau)dv d\tau$$
$$= \int_0^t \int_{\mathbb{R}^3} Q(f,f)(\tau,v)\phi(\tau,v)\,dv\,d\tau.$$
$$= \frac{1}{2}\int_0^t \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} \underbrace{B(v-v_*,w)}_{-3<\gamma<-2}[\phi_*' + \phi' - \phi - \phi_*]\,dw\,dv_*\,dv\,d\tau. \quad \boxed{<\infty}$$

if f satisfies (a) and $\phi \in W^{2,\infty}(\mathbb{R}^3)$ $\qquad \sim |v-v_*|^2$

(d') $\forall \phi \in C_c^\infty(\mathbb{R}^3)$, $\forall s, t > 0$

$$\int_{\mathbb{R}^3} f(t,v)\phi(v)dv - \int_{\mathbb{R}^3} f(s)\phi(v)dv = \int_s^t \int_{\mathbb{R}^3} Q(f,f)(\tau,v)\,dv\,d\tau$$

Definition 2 ("weak" H-solution for $-3<\gamma<-2$) Villani 98

Let $f_0(v)$ be initial condition with finite mass, energy and entropy. We say $f(t,v)$ is (weak) H-solution to Cauchy Problem(*) if f satisfies (a)-(d), except the last integration being defined by:

$$\boxed{\int_{\mathbb{R}^3} Q(f,f)(v)\,\phi(v)\,dv}$$

$-3<\gamma<-2$

$$= \int_{\mathbb{R}^3}\int_{S^2} b(\cos\theta)\,|v-v_*|^\gamma\,(f'f_*'-ff_*)(\phi_*'+\phi'-\phi-\phi_*)\,d\sigma\,dv_*\,dv.$$

$$= \int_{\mathbb{R}^3}\int_{S^2} b(\cos\theta)\,\frac{F(x')-F(x)}{|x_2|^2}\,[\phi_*'+\phi'-\phi-\phi_*]\,d\sigma\,dv_*\,dv.$$

$$= \int_{\mathbb{R}^3}\int_{S^2} \frac{b(\cos\theta)}{|x|^2}\,(\sqrt{F'}-\sqrt{F})(\sqrt{F'}+\sqrt{F})[\phi_*'+\phi'-\phi-\phi_*]\,d\sigma\,dv_*\,dv$$

$$= 2\pi\int_{\mathbb{R}^3}\int_0^{\frac{\pi}{2}}\left(\sqrt{b(\cos\theta)}\,\sin\theta\,\frac{\sqrt{F'}-\sqrt{F}}{|x_2-x_2'|}\right)\left[\sqrt{b(\cos\theta)}\,\frac{\sqrt{F'}+\sqrt{F}}{|x_2|}\right][\phi_*'+\phi'-\phi-\phi_*]\,d\sigma\,dv_*\,dv.$$

$$\leq \left\|\sqrt{b(\cos\theta)}\,\sin\theta\,\frac{\sqrt{F'}-\sqrt{F}}{|x_2-x_2'|}\right\|_{L^2(dtdv d\theta d\phi dv_*)} \qquad \in L^2 \qquad \sim O(|x_2|^2\theta)$$

$<\infty$

$\sim \sqrt{ff_*}\,|v-v_*|^{\frac{\gamma}{2}+2}\,\theta\sqrt{b(\cos\theta)}$

$\sim \sqrt{ff_*}\,(1+|v|+|v_*|) \in L^2(dv dv_*)$

$\leq C$

Pf: Select $\boxed{\phi=\ln f}$

$$\int_0^T\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} b(\cos\theta)|v-v_*|^\gamma\,[f'f_*'-ff_*](\ln f_*'+\ln f'-\ln f-\ln f_*)$$

Consider $\boxed{|v'-v_*'|=|v-v_*|} \Leftrightarrow |v'-v_*'|^\gamma=|v-v_*|^\gamma$ $\qquad \ln f_*'f'-\ln ff_*$

$$\Rightarrow \ln f'f_*'-\ln ff_* = \ln f'f_*'\,|v'-v_*'|^{\gamma+2}-\ln ff_*|v-v_*|^{\gamma+2}$$

$\leq C$

$$\int_0^T\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} b(\cos\theta)\frac{(f'f_*'\,|v'-v_*'|^{\gamma+2}-ff_*|v-v_*|^{\gamma+2})(\ln f'f_*'|v'-v_*'|^{\gamma+2}-\ln ff_*|v-v_*|^{\gamma+2})}{|v-v_*|^2}$$

Set $F(x_1,x_2)=F(v,v_*)=ff_*|v-v_*|^{\gamma+2}$

$$\begin{cases} v'=\dfrac{v+v_*}{2}+\dfrac{|v-v_*|}{2}\sigma \\ v_*'=\dfrac{v+v_*}{2}-\dfrac{|v-v_*|}{2}\sigma \end{cases} \Rightarrow x'=(x_1,|x_2|\sigma)$$

$x_1 \Updownarrow v+v_*$, $\quad x_2 \Downarrow v-v_*$

$$\Rightarrow |x_2 - x_2'| = |v - v_*| \left| \frac{v-v_*}{|v-v_*|} - \sigma \right| = 2|v-v_*| \sin\frac{\theta}{2}$$

Note the classical inequlity:

$(\ln x - \ln y)(x-y) \geq 4|\sqrt{x} - \sqrt{y}|^2$ valid for $x, y \geq 0$.

$\Rightarrow$ entropy production estimate:

$$\int_0^T \int_0^{2\pi} \int_0^{\frac{\pi}{2}} b(\cos\theta) \sin^2\frac{\theta}{2} \, d\theta d\phi \int \frac{|\sqrt{F(x')} - \sqrt{F(x)}|^2}{|x_2 - x_2'|^2} dx \leq C$$

$\sigma \mapsto (\theta, \phi)$ $\sin\theta$

or equavalently, $\boxed{\sqrt{b(\cos\theta)} \sin\frac{\theta}{2} \frac{|\sqrt{F(x')} - \sqrt{F(x)}|}{|x_2 - x_2'|^2}} \in L^2(dt d\theta d\phi dx)$

For Definition 1 ($\underset{\text{hard potential}}{\gamma > 0}$, $\underset{\text{moderately soft potential.}}{-2 < \gamma < 0}$).

$\phi \in W^{2,\infty}$

$$\int Q(f,f)\phi \, dv = \frac{1}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{S^2} B(v-v_*, w) f f_* [\phi_*' + \phi' - \phi_* - \phi]$$

$B(v-v_*,w) \Downarrow \boxed{\sin\theta \, b(\cos\theta)} \, |v-v_*|^\gamma$, $\sin\theta \, b(\cos\theta) \sim \theta^{-1-\nu}$, $0 < \nu < 2$

$[\phi_*' + \phi' - \phi_* - \phi] \sim O(\theta^2 |v-v_*|^2)$

Taylor Expansion.

$$\phi(v') - \phi(v) = \nabla\phi \cdot (v'-v) + \int_0^1 D^2\phi[v + t(v'-v)] : (v'-v) \otimes (v'-v)(1-t) dt.$$

Recall $\boxed{v' = v - (v-v_*, w)w}$

$\Rightarrow v' - v = -(v-v_*, w)w$

$$= -(v-v_*, w)(\nabla\phi, w) + (v-v_*, w)^2 \int_0^1 D^2\phi(v - t(v-v_*, w)w)(w,w)(1-t)dt.$$

$$\phi(v_*') - \phi(v_*) = (v-v_*, w)(\nabla_*\phi_*, w) + (v-v_*, w)^2 \int_0^1 D^2\phi(v + t(v-v_*, w)w)(w,w)(1-t)dt.$$

$\boxed{\phi(v_*') + \phi(v') - \phi(v_*) - \phi(v)}$

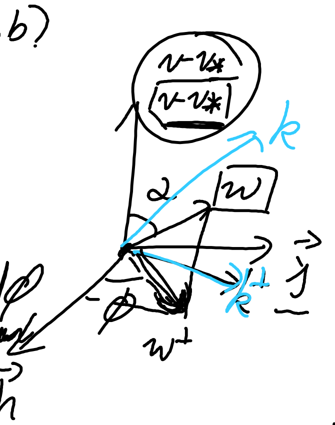
$$= [(\nabla\phi - \nabla_*\phi_*), w](v - v_*, w) + (v-v_*, w)^2 \int_0^1 D^2\phi(v - t(v-v_*, w)w)$$

$\Downarrow O(|v-v_*|)$ $\quad |(v-v_*), w| = |v-v_*| \sin\frac{\theta}{2}$ $\quad (w,w)(1-t)dt$

$D^2\phi(v)|v-v_*|)$

$(k, w)(v-v_*, w)$

$\sim O(|v-v_*|\,\theta \wedge 1)$, $\quad a\wedge b = \min(a,b)$

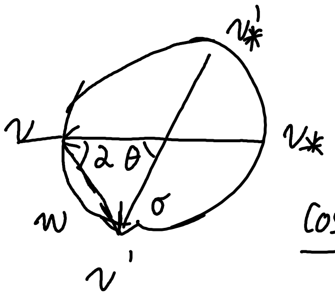


$$\int_{S^2} B(v-v_*, w)\,[\phi_*' + \phi' - \phi_* - \phi]\,dw$$

$$= \int_0^{2\pi}\int_0^{\pi} b(\cos\theta)\sin\theta\,|v-v_*|^{\gamma}\,[\phi_*' + \phi' - \phi_* - \phi]\,d\theta\,d\phi$$

For the average of $[\phi_*' + \phi' - \phi_* - \phi]$ for all the value of polar angle ϕ:

Set $k = \nabla\phi - \nabla_*\phi_*$, $\quad k = \left(k, \frac{v-v_*}{|v-v_*|}\right)\frac{v-v_*}{|v-v_*|} + k^{\perp}$



$2\alpha + \theta = \pi$

$\Rightarrow \alpha = \frac{\pi - \theta}{2}$

$\cos\alpha = \cos\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = \sin\frac{\theta}{2}$

$$w = \frac{v-v_*}{|v-v_*|}\cos\alpha + (\cos\phi\,\vec{h} + \sin\phi\,\vec{j})\sin\alpha$$

$$= \boxed{\frac{v-v_*}{|v-v_*|}\sin\frac{\theta}{2}} + \underbrace{\boxed{(\cos\phi\,\vec{h} + \sin\phi\,\vec{j})\cos\frac{\theta}{2}}}_{w^{\perp}}$$

$$(k^{\perp}, w) \sim A\cos\phi + B\sin\phi \Rightarrow \int_0^{2\pi}(k^{\perp}, w)\,d\phi = 0$$

$$-\int_0^{2\pi}(v-v_*, w)(k, w)\,d\phi = -\int_0^{2\pi}(v-v_*, w)\left(\left[\left(k, \frac{v-v_*}{|v-v_*|}\right)\frac{v-v_*}{|v-v_*|} + k^{\perp}\right], w\right)d\phi$$

$(v-v_*, w)$

$\Downarrow$

$|v-v_*|\,|w|\cdot\cos\alpha$

$= |v-v_*|\sin\frac{\theta}{2}$

$$= -|v-v_*|\sin\frac{\theta}{2}\int_0^{2\pi}\left(\left[\left(k, \frac{v-v_*}{|v-v_*|}\right)\frac{v-v_*}{|v-v_*|}, \frac{v-v_*}{|v-v_*|}\sin\frac{\theta}{2} + w^{\perp}\right]\right)d\phi$$

$$= -\cancel{|v-v_*|}\sin\frac{\theta}{2}\int_0^{2\pi}\sin\frac{\theta}{2}\left(k, \frac{v-v_*}{\cancel{|v-v_*|}}\right)d\phi$$

$$= -2\pi\sin^2\frac{\theta}{2}\,(k, v-v_*)$$

$$v\varphi - v_*\varphi \sim O(|v-v_*|)$$

$$\approx -2\pi \underbrace{\sin^2\frac{\theta}{2}}\, |v-v_*|^2$$

$$\Longrightarrow \int_0^{2\pi} [\underbrace{\phi_*' + \phi' - \phi - \phi_*}] \, d\phi = O(\theta^2 \underbrace{|v-v_*|^2})$$

$$\Longrightarrow \underline{\int_{K \subset \mathbb{R}^3 \times \mathbb{R}^3} |v-v_*|^\gamma f f_* [\quad] \, dv dv_*} = \int_K f f_* \underline{|v-v_*|^{\gamma+2}} < \infty.$$

$\gamma > 0$,

$-2 < \gamma < 0$.

Prove Well-posedness (Look for solution)

$\{f^n\}$ $\boxed{|v-v_*| \chi_{|v-v_*| \le n}}$

We construct sequence of collision kernel

$b(\cos\theta)$

$\theta \sim 0$ singularity

$$\underline{B_n}(|v-v_*|, \theta) = \underset{\Downarrow}{b_n(\cos\theta)}\, |v-v_*|^\gamma \, \chi_{|v-v_*| \le n}$$

$$b(\cos\theta)\chi_{\{\theta \ge \frac{1}{n}\}}$$

Exercise 7

Remove cutoff for soft-potential case.

$\theta \chi_{\{\theta \ge \frac{1}{n}\}} \to \theta$, as $n \to \infty$.

$$\begin{cases} \partial_t f^n = Q_n(f^n, f^n) \\ f(t=0, v) = f_0(v) \end{cases}$$

$$Q_n(f^n,f^n) = \int_{\mathbb{R}^3}\int_{S^2} B_n(|v-v_*|, \sigma)[f_*'f' - f_* f] \, d\sigma dv_*$$

$\overset{\text{lecture 2}}{\Longrightarrow}$ $\{f^n\}$ preserves mass, energy. Entropy decrease.

$\Longrightarrow$ $\{f^n\} \in L^1_2 \cap L\log L$ Uniformly bounded.

Equi-continuous in t: (Moderately soft potential $-2 < \gamma < 0$)

$$\left| \int_{\mathbb{R}^3} f^n(t)\phi(v)\,dv - \int_{\mathbb{R}^3} f^n(s)\phi(v)\,dv \right| \le \int_s^t \left| \int_{\mathbb{R}^3} Q_n(f^n, f^n)(\tau)\phi \, dv \right| d\tau$$

$-2 < \gamma < 0$

$0 < 2+\gamma < 2$

$$\le |t-s| C_\phi \int_{\mathbb{R}^3\times\mathbb{R}^3} (1 + |v-v_*|^{\gamma+2}) f f_* \, dv dv_*$$

$$\le |t-s| C_\phi \int_{\mathbb{R}^3\times\mathbb{R}^3} (1 + |v|^{\gamma+2} + |v_*|^{\gamma+2}) f f_* \, dv dv_*$$

$$\le |t-s| C_\phi \int (1+|v|^{\gamma+2})(1+|v_*|^{\gamma+2}) f f_* \, dv dv_*$$

$$\le C_\phi |t-s| \, \|f^n\|^2_{L^1_{\gamma+2}}$$

(Hard potential $0<\gamma<1$)

$$\left| \int_{\mathbb{R}^3} f^n(t)\phi(v)\,dv - \int_{\mathbb{R}^3} f^n(s)\phi(v)\,dv \right|$$

$$\le |t-s|\, C_\phi \int_{\mathbb{R}^3\times\mathbb{R}^3} f^n(v) f^n(v_*)\left(1+|v|^{\gamma+1}+|v_*|^{\gamma+1}\right) dv\,dv_*$$

$$\le C_\phi |t-s| \int_{\mathbb{R}^3\times\mathbb{R}^3} f^n(v) f^n(v_*)\left(1+|v|^{\gamma+1}\right)\left(1+|v_*|^{\gamma+1}\right) dv\,dv_*$$

$$\le C_\phi |t-s| \underbrace{\|f^n\|^2_{L^1_{\gamma+1}}}_{} \qquad \boxed{0<\gamma<1}$$

Notice energy estimate $\|\cdot\|_{L^1_2}$ is enough if $0<\gamma<1$.

Recall that $\gamma>0 \Rightarrow \gamma = \frac{l-5}{l-1} \Rightarrow l>5 \Rightarrow \nu = \frac{2}{l-1}$

$\theta \to \theta$ $\quad \frac{1}{v^{l-1}}$ $\qquad \Downarrow$

$\nu<\frac{1}{2}$

Remark: $\quad \sin^{d-2}\theta\, b(\cos\theta)\big|_{\theta\to 0^+} \sim K\theta^{-1-\nu}$

Energy finite is enough if $0<\nu<\frac{1}{2}$. $\quad \sim K\theta^{-\frac{3}{2}}$

But if we want to handle more singular kernel, we have to apply $f_0 \in L^1_{2+k}$, $k>0$ #. $\quad$ Full range $\boxed{0<\nu<2}$