Refined Povzner Inequality and Moment Gain Property

We start with the case of the initial datum f_0 with finite "$2+k$" moments, for some $k>0$. $\|f_0\|_{L^1_{2+k}}$

Theorem: Assume that f_0 satisfies $\langle v\rangle = (1+|v|^2)^{\frac{1}{2}}$

$$\boxed{\int_{\mathbb{R}^3} \langle v\rangle^{2+k} f_0(v)\,dv < \infty, \text{ for some } k>0}$$ Finite "$2+k$" moments.

Let $\{f^n\}$ and f obtained in last lecture_3.

$$(*)\begin{cases} \partial_t f^n = Q^n(f^n,f^n) \\ f_n(t=0,v) = f_0(v) \end{cases} \longrightarrow (**)\begin{cases} \partial_t f = Q(f,f) \\ f(t=0,v) = f_0(v) \end{cases}$$

$$b_n |v-v_*|^\gamma \chi_{\{|v-v_*|>n\}} \longrightarrow b(\cos\theta)|v-v_*|^\gamma, \begin{cases} 0<\gamma<2 \text{ Hard} \\ -2<\gamma<0 \text{ soft.} \end{cases}$$

$$b(\cos\theta)\chi_{\{\theta\geq\frac{1}{n}\}}$$

$$0<\gamma<2.$$

$$\sin\theta\, b(\cos\theta)\big|_{\theta\to 0^+} \sim K\theta^{-1-\gamma}$$

Then, for any fixed time $T>0$, there exist a $C_T>0$ independent of "n" such that:

(i) For the soft potential $(-2\leq\gamma<0)$ Preserve the moments.

$$\sup_{t\in[0,T]}\int_{\mathbb{R}^3}\langle v\rangle^{2+k}\boxed{f^n_t(v)}\,dv + \sup_{t\in[0,T]}\int_{\mathbb{R}^3}\langle v\rangle^{2+k}\boxed{f_t(v)}\,dv \leq C_T.$$

$$f^n_t(v) = f^n(t,v)$$

(ii) For the hard potential $(0<\gamma\leq 2)$

$$\sup_{t\in[0,T]}\int_{\mathbb{R}^3}\langle v\rangle^{2+k} f^n_t(v)\,dv + \int_0^T\int_{\mathbb{R}^3}\langle v\rangle^{2+k}\underbrace{\min\{\langle v\rangle^\gamma, n^\gamma\}}_{\downarrow n\to\infty} f^n_\tau(v)\,dv\,d\tau \leq C_T$$

$$\sup_{t\in[0,T]}\int_{\mathbb{R}^3}\langle v\rangle^{2+k} f_t(v)\,dv + \int_0^T\int_{\mathbb{R}^3}\langle v\rangle^{2+k+\gamma} f_\tau(v)\,dv\,d\tau \leq C_T.$$

Pf: $\boxed{\text{Step 1:}}$ Prove $\int_{\mathbb{R}^3} \langle v\rangle^\beta f_t^n(v)\,dv < \infty$, if $\int_{\mathbb{R}^3} f_0(v)\langle v\rangle^\beta dv < \infty$ (Lemma 1)

where $f_t^n(v)$ is the solution to $(*)$

Define $W_\delta(v) = \langle v\rangle^\beta \langle \delta v\rangle^{-\beta}$ for some $\delta > 0$.

$$W_\delta(v') = \left(\frac{1+|v'|^2}{1+\delta^2|v'|^2}\right)^{\frac{\beta}{2}} \le \left(\frac{1+|v|^2+|v_*|^2}{1+\delta(|v|^2+|v_*|^2)}\right)^{\frac{\beta}{2}}$$

$$\le \left(\frac{1+|v|^2}{1+\delta^2|v|^2} + \frac{1+|v_*|^2}{1+\delta^2|v_*|^2}\right)^{\frac{\beta}{2}}$$

$$\lesssim W_\delta(v) + W_\delta(v_*).$$

Select $\phi(v) = W_\delta(v)$

$$\frac{d}{dt}\int_{\mathbb{R}^3} W_\delta(v) f_t^n(v)\,dv = \int_{\mathbb{R}^3\times\mathbb{R}^3}\int_{S^2} b_n(\cos\theta)\,\underset{\Uparrow}{\Phi_n(|v-v_*|)}\, W_\delta(v)\,(f_t^n(v')f_t^n(v_*') - f^n f_*^n)\,d\sigma\,dv_*\,dv$$

$|v-v_*|^\gamma \chi_{|v-v_*|\ge n}$

$$\le \int_{\mathbb{R}^3\times\mathbb{R}^3}\int_{S^2} b_n \Phi_n (W_\delta(v')) f_*^n f^n\,dv_*\,dv$$

$$\le \int_{\mathbb{R}^3\times\mathbb{R}^3}\int_{S^2} b_n \Phi_n (W_\delta(v) + W_\delta(v_*)) f_*^n f^n\,dv\,dv_*$$

$$\lesssim \int_{\mathbb{R}^3} W_\delta(v) f_t^n(v)\,dv.$$

$\Longrightarrow \int_{\mathbb{R}^3} f_t^n(v) W_\delta(v)\,dv < C_T$, for $t\in[0,T]$

$\overset{\delta\to 0}{\Longrightarrow} \int_{\mathbb{R}^3} f_t^n(v)\langle v\rangle^\beta dv < \infty.$

If we set $\boxed{W_\delta(v) = \langle v\rangle^{2+\kappa}\langle \delta v\rangle^{-2-\kappa}}$, by Lemma 1 above and $\delta\to 0$.

$\Longrightarrow \int_{\mathbb{R}^3} \langle v\rangle^{2+\kappa} f_t^n(v)\,dv < \infty$, for any $t > 0$.

Step 2: Refined Povzner Inequality (Cutoff and Non-Cutoff)

$\int f\langle v\rangle^{2+\kappa} dv$

$\Psi(v) = \bar{\Psi}(|v|^2)$ $\qquad \in W^{2,\infty}$

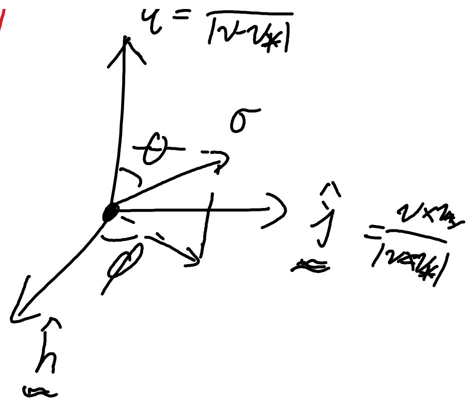
$$\frac{d}{dt}\int f\Psi\,dv = \frac{1}{2}\iiint_{S^2} b_n \Phi_n f f_* [\Psi(v_*') + \Psi(v') - \Psi(v) - \Psi(v_*)]\,d\sigma\,dv\,dv_*$$

Since $\sigma\in S^2$, it can be de-composed as

$$\boxed{\sigma = \hat{q}\cos\theta + \sin\theta\cos\phi\,\hat{h} + \sin\theta\sin\phi\,\hat{j}}$$

$\theta\in[0,\pi]$, $\phi\in[-\pi,\pi]$

by orthogonal basis in $\mathbb{R}^3$.



$$\begin{cases} \hat{q} = \dfrac{v-v_*}{|v-v_*|} \\ \hat{j} = \dfrac{v\times v_*}{|v\times v_*|} \\ \hat{h} = \dfrac{((v-v_*)\cdot v)v_* - ((v-v_*)\cdot v_*)v}{|v-v_*|\,|v\times v_*|} \end{cases}$$

$$\Rightarrow \hat{j}\perp(v+v_*) \quad \text{and} \quad v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\underset{\text{unit vector}}{\sigma}$$

$$|v'|^2 = \left|\frac{v+v_*}{2}\right|^2 + \left|\frac{v-v_*}{2}\right|^2 + \frac{|v-v_*|}{2}(v+v_*)\cdot\sigma$$

$$= \frac{|v|^2+2v\cdot v_*+|v_*|^2}{4} + \frac{|v|^2-2v\cdot v_*+|v_*|^2}{4} + \frac{|v-v_*|}{2}\left((v+v_*)\cdot\left(\cos\theta\underbrace{\hat{q}}_{\frac{v-v_*}{|v-v_*|}} + \sin\theta\cos\phi\,\hat{h}\right)\right)$$

$$= \frac{\cancel{2}|v|^2+\cancel{2}|v_*|^2}{\cancel{4}\,2} + \frac{\cos\theta}{2}(|v|^2-|v_*|^2) + \frac{\sin\theta\cos\phi}{2|v\times v_*|}\left[(v+v_*)\cdot\left(((v-v_*)\cdot v)v_* - ((v-v_*)\cdot v_*)v\right)\right]$$

$$= \frac{(1+\cos\theta)|v|^2}{2} + \frac{(1-\cos\theta)|v_*|^2}{2} + |v||v_*|\sin\alpha\sin\theta\cos\phi$$

α is angle between v and v_*.

$$\Rightarrow |v'|^2 = \underbrace{|v|^2\cos^2\frac{\theta}{2} + |v_*|^2\sin^2\frac{\theta}{2}}_{Y(\theta)} + \underbrace{|v\times v_*|\sin\theta}_{\boxed{Z(\theta)}}\,\underbrace{\cos\phi}_{\cos\phi}$$

$$= Y(\theta) + Z(\theta)\cos\phi$$

$|Z(\theta)| = |v\times v_*|\sin\theta$

$= |(v-v_*)\times v_*|\sin\theta$

$\le |v-v_*|\,|v_*|\sin\theta \sim \theta$

$\theta^{-2<\gamma<0}$ $|v-v_*|^{\gamma}$

On the other hand,

$$|v_*'|^2 = \underbrace{|v_*|^2\cos^2\frac{\theta}{2} + |v|^2\sin^2\frac{\theta}{2}}_{=Y(\pi-\theta)} - \underbrace{|v\times v_*|\sin\theta}_{Z(\theta)}\,\underbrace{\cos\phi}_{\cos\phi}.$$

$$= Y(\pi-\theta) - \boxed{Z(\theta)}\cos\phi$$

$|v-v_*|^2$

$\boxed{|Z|^2}$

Cutoff

Non-cutoff

② $\sin\theta\, b|_{\theta\to 0} \sim \theta^{-1-2}$ $\quad 0<2\nu<2$

$\theta^2 \sin\theta\, b \sim \theta^{-1}$ ✓

If we set $\Psi(x) = \Psi_k(x) = (1+x)^{1+\frac{k}{2}}$

$$= [(1+x)^{\frac{1}{2}}]^{2+k}$$

$$\Rightarrow \Psi_1(|v|^2) = [(1+|v|^2)^{\frac{1}{2}}]^{2+k} = \langle v\rangle^{2+k}.$$

$$\Psi_2(x) = \Psi_k(x) = x^{1+\frac{k}{2}}$$

$$\Rightarrow \Psi_2(|v|^2) = (|v|^2)^{1+\frac{k}{2}} = |v|^{2+k}$$

① Convex function. ✓

② $\Psi_{1,2}(x) \geq 0,\ x>0$

③ $\Psi_{1,2} \in C^2([0,\infty))$

Select Ψ_1, by weak solution. - ...

$$\frac{d}{dt}\int_{\mathbb{R}^3} \langle v\rangle^{2+k} f_t^n(v)\,dv = \frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3} \Phi_n K_n(v-v_*) f_t^n(v) f_t^n(v_*)\,dv\,dv_*$$

$$\Rightarrow \text{where } \boxed{K_n(v,v_*)} = \int_{S^2} b_n(\cos\theta)[\Psi(|v'|^2)+\Psi(|v_*'|^2)-\Psi(|v|^2)-\Psi(|v_*|^2)]\,d\sigma$$

$\sigma \to (\theta,\phi)$

$$= \int_0^{2\pi}\int_0^{\pi} b_n(\cos\theta)[\Psi(|v'|^2)+\Psi(|v_*'|^2)-\Psi(|v|^2)-\Psi(|v_*|^2)]$$

$$\Rightarrow \int_0^{\pi} b_n(\cos\theta)\int_0^{\pi}[\underbrace{\Psi(|v'|^2)}_{\sim\cos\phi}+\underbrace{\Psi(|v_*'|^2)}_{\sim\cos\phi}-\Psi(|v|^2)-\Psi(|v_*|^2)]\,d\phi \quad \sin\theta\, d\theta\, d\phi$$

$\pi\Psi(Y)$ ② $\quad \pi\Psi(Y(\pi-\theta))$ ② $\quad \pi\Psi(|v|^2) \quad \sin\theta\, d\theta \quad \pi\Psi(|v_*|^2)$

Note that

$$\boxed{\int_0^{\pi}\Psi(|v'|^2)\,d\phi} = \int_0^{\pi}\Psi(Y(\theta)+Z(\theta)\cos\phi)\,d\phi$$

$$= \left(\int_0^{\frac{\pi}{2}}+\int_{\frac{\pi}{2}}^{\pi}\right)\Psi(Y(\theta)+Z(\theta)\cos\phi)\,d\phi$$

$\phi \to \pi-\phi$

$$= \int_0^{\frac{\pi}{2}} \Psi(Y(\theta)+Z(\theta)\cos\phi)+\Psi(Y(\theta)-Z(\theta)\cos\phi)\,d\phi.$$

$$= \int_0^{\frac{\pi}{2}} \Psi(Y(\theta)+Z(\theta)\cos\phi)+\Psi(Y(\theta)-Z(\theta)\cos\phi)-2\Psi(Y(\theta))\,d\phi$$

$$+\pi\Psi(Y(\theta))$$

Integration by part $\Rightarrow \pi\Psi(Y)+\{\phi[\Psi(Y+Z\cos\phi)+\Psi(Y-Z\cos\phi)-2\Psi(Y)]\}\Big|_0^{\frac{\pi}{2}}$

$\int^{\frac{\pi}{2}} \ldots$

$$-\int_0 \varphi\left[\Psi'(Y+z\cos\phi)(-\sin\phi z)+\Psi'(Y-z\cos\phi)(\sin\phi z)\right]d\phi$$

$$\underbrace{\cancel{\frac{\pi}{2}\left[\Psi(Y)+\Psi(Y)-2\Psi(Y)\right]}}_{0}$$

$$\Longrightarrow \pi\Psi(Y)+z\int_0^{\frac{\pi}{2}}\phi\sin\phi\left[\Psi'(Y+z\cos\phi)-\Psi'(Y-z\cos\phi)\right]d\phi$$

$\xrightarrow[\text{by part again}]{\text{Integration}}$ $\pi\Psi(Y)+z\int_0^{\frac{\pi}{2}}\left\{\left[\Psi'(Y+z\cos\phi)-\Psi'(Y-z\cos\phi)\right]\right\}d(\sin\phi-\phi\cos\phi)$

$$\Longrightarrow \pi\Psi(Y)+\cancel{\left\{z(\sin\phi-\phi\cos\phi)\left[\Psi'(Y+z\cos\phi)-\Psi'(Y-z\cos\phi)\right]\right\}\Big|_0^{\frac{\pi}{2}}}$$

$$-z\int_0^{\frac{\pi}{2}}(\sin\phi-\phi\cos\phi)\left[\Psi''(Y+z\cos\phi)(-z\sin\phi)-\Psi''(Y-z\cos\phi)(z\sin\phi)\right]d\phi$$

when $\phi=\frac{\pi}{2}$, $\cos\frac{\pi}{2}=0$, $\Psi'(Y)-\Psi'(Y)=0$

when $\phi=0$, $\sin\phi=\phi=0$

$$\Longrightarrow \pi\Psi(Y)+z^2\int_0^{\frac{\pi}{2}}(\sin\phi-\phi\cos\phi)\sin\phi\left[\Psi''(Y+z\cos\phi)+\Psi''(Y-z\cos\phi)\right]d\phi.\checkmark$$

$$\int_0^{\pi}\Psi(|v_*|^2)\,d\phi=\pi\Psi(Y(\pi-\theta))$$

$$+z^2\int_0^{\frac{\pi}{2}}(\sin\phi-\phi\cos\phi)\sin\phi\left[\Psi''(Y(\pi-\theta)+z\cos\phi)+\Psi''(Y(\pi-\theta)+z\cos\phi)\right]d\phi$$

Recall that:

$$K_n(v,v_*)=2\int_0^{\pi}\int_0^{\pi}b_n(\cos\theta)\left[\Psi(|v'|^2)+\Psi(|v_*'|^2)-\Psi(|v|^2)-\Psi(|v_*|^2)\right]d\phi\,\sin\theta\,d\theta.$$

$\pi\Psi(Y)$ $\quad z^2\int\square\,d\phi$ $\quad \pi\Psi(Y(\pi-\theta))$ $\quad z^2\int\square\,d\phi$

$$=\boxed{-H_n(v,v_*)}+G_n(v,v_*)$$

$t_1+t_2=1$ convexity

$\Psi(t_1x_1+t_2x_2)\le t_1\Psi(x_1)+t_2\Psi(x_2)$

$$-H_n(v,v_*)=2\pi\int_0^{\pi}b_n(\cos\theta)\sin\theta\left[\Psi(Y(\theta))+\Psi(Y(\pi-\theta))-\Psi(|v|^2)-\Psi(|v_*|^2)\right]d\theta$$

$$=2\pi\int_0^{\pi} b_n(\cos\theta)\sin\theta\Big[\underline{\Psi(|v|^2\boxed{\cos^2\tfrac{\theta}{2}}+|v_*|^2\boxed{\sin^2\tfrac{\theta}{2}})}+\Psi(|v|^2\boxed{\sin^2\tfrac{\theta}{2}}+|v_*|^2\boxed{\cos^2\tfrac{\theta}{2}})$$

$$-(\boxed{\cos^2\tfrac{\theta}{2}}+\boxed{\sin^2\tfrac{\theta}{2}})\Psi(|v|^2)-(\boxed{\cos^2\tfrac{\theta}{2}}+\boxed{\sin^2\tfrac{\theta}{2}})\Psi(|v_*|^2)\Big]d\theta.$$

$$=2\pi\int_0^{\pi} b_n(\cos\theta)\sin\theta\Big[\underbrace{\Psi(|v|^2\cos^2\tfrac{\theta}{2}+|v_*|^2\sin^2\tfrac{\theta}{2})-\Big(\cos^2\tfrac{\theta}{2}\Psi(|v|^2)+\sin^2\tfrac{\theta}{2}\Psi(|v_*|^2)\Big)}_{\leq 0}$$

$$+\Big[\underbrace{\Psi(|v|^2\sin^2\tfrac{\theta}{2}+|v_*|^2\cos^2\tfrac{\theta}{2})-\Big(\sin^2\tfrac{\theta}{2}\Psi(|v|^2)+\cos^2\tfrac{\theta}{2}\Psi(|v_*|^2)\Big)}_{\leq 0}\Big]d\theta.$$

$$\leq 0. \qquad \leq \boxed{\langle v\rangle 1_{\langle v\rangle\geq\langle v_*\rangle}+\langle v_*\rangle 1_{\langle v_*\rangle\geq\langle v\rangle}}$$

$$G_n(v,v_*)=2\int_0^{\pi} b_n(\cos\theta)\sin\theta\; z^2(\theta)\int_0^{\frac{\pi}{2}}(\sin\phi-\phi\cos\phi)\sin\phi\Big[\Psi''(Y+z\cos\phi)+\Psi''(Y-z\cos\phi)+\Psi''(Y(\pi-\theta)+z\cos\phi)+\Psi''(Y(\pi-\theta)-z\cos\phi)\Big]d\phi\, d\theta$$

$\Rightarrow$ Let's first estimate

$$z^2\int_0^{\frac{\pi}{2}}\underbrace{(\sin\phi-\phi\cos\phi)\sin\phi}_{\sim\phi^3}\Big[\Psi''(Y+z\cos\phi)+\Psi''(Y-z\cos\phi)\Big]d\phi.$$

$$\Psi(x)=(1+x)^{1+\frac{k}{2}}$$

$$\Rightarrow \Psi'(x)=(1+\tfrac{k}{2})(1+x)^{\frac{k}{2}}$$

$$\Rightarrow \Psi''(x)=(1+\tfrac{k}{2})\tfrac{k}{2}(1+x)^{\frac{k}{2}-1}$$

$$=C_{(k)}\,z^2\int_0^{\frac{\pi}{2}}\phi^3\Big[(1+Y+z\cos\phi)^{\frac{k}{2}-1}+(1+Y-z\cos\phi)^{\frac{k}{2}-1}\Big]d\phi.$$

$$z_0=\frac{z(\theta)}{1+Y(\theta)}\in[0,1] \qquad \frac{Y+z\cos\phi}{1+Y}=\underbrace{\frac{Y}{1+Y}}_{\leq 1}+\underbrace{\frac{z\cos\phi}{1+Y}}_{z_0}$$

$$=C_{(k)}\,z^2(1+Y)^{\frac{k}{2}-1}\int_0^{\frac{\pi}{2}}\phi^3\Big[(1+\underbrace{z_0\cos\phi}_{0\leq\,\leq 1})^{\frac{k}{2}-1}+(1-\underbrace{z_0\cos\phi}_{0\leq\,\leq 1})^{\frac{k}{2}-1}\Big]d\phi$$

$$=\begin{cases}\text{if } k<2 \Leftrightarrow \frac{k}{2}-1<0, & \lesssim z^2\leq |v|^2|v_*|^2\theta^2\\[2ex] \text{if } k\geq 2\Leftrightarrow \frac{k}{2}-1\geq 0, & \lesssim z^2(1+Y)^{\frac{k}{2}-1}\leq |v|^2|v_*|^2\theta^2\left(\langle v\rangle^{k-2}+\langle v_*\rangle^{k-2}\right)\end{cases}$$

$$1+Y=1+\cos^2\tfrac{\theta}{2}|v|^2+\sin^2\tfrac{\theta}{2}|v_*|^2$$

$$\leq\Big((1+|v|^2)^{\frac{k}{2}-1}+(1+|v_*|^2)^{\frac{k}{2}-1}\Big)$$

Consequently, there exists C_0, C_1 independant of "n" such that

$$G_n(v,v_*)\ \Big\{\leq C_0|v|^2|v_*|^2\int_0^{\pi}b_n(\cos\theta)\sin\theta\,\theta^2 d\theta\leq C_1|v|^2|v_*|^2, \text{ if } k<2$$

$$I \le C_0 |v|^2 |v_*|^2 (\langle v\rangle^{k-2} + \langle v_*\rangle^{k-2}) \int_0^{\pi} b_n(\cos\theta)\sin\theta\, \theta^2 d\theta \le C_1 (|v_*|^2 \langle v\rangle^k + |v|^2 \langle v_*\rangle^k)$$

if $k \ge 2$

For the soft-potential case, $-2 < \gamma < 0$.

$$\int_{\mathbb{R}^3} \langle v\rangle^{2+k} f_t^n(v)\,dv \le \int_{\mathbb{R}^3} \langle v\rangle^{2+k} f_0(v)\,dv + C\left(\int_{\mathbb{R}^3} \langle v\rangle f_0(v)\,dv\right) \int_0^t \int_{\mathbb{R}^3} \langle v\rangle^{\max\{k,2\}} f_\tau^n(v)\,dv\,d\tau$$

For the hard-potential case, $\gamma > 0$.

We need more detailed estimate for $H_n(v, v_*)$:

$$\boxed{H_n(v,v_*)} = \int_0^{\frac{\pi}{2}} b_n(\cos\theta)\sin\theta \left[\Psi(|v|^2) + \Psi(|v_*|^2) - \Psi(Y) - \Psi(Y(\pi-\theta))\right] d\theta$$

there exist $[\theta_1, \theta_2] \subset (0, \frac{\pi}{2})$ and constant C_0 indepent of "n",

$b_n(\cos\theta)\sin\theta \ge C_0$

$$\ge 2\pi C_0 (\theta_2 - \theta_1) \left[\Psi(|v|^2) + \Psi(|v_*|^2) - \Psi(|v|^2\cos^2\tfrac{\theta}{2} + |v_*|^2 \sin^2\tfrac{\theta}{2}) - \Psi(|v_*|^2\cos^2\tfrac{\theta}{2} + |v|^2\sin^2\tfrac{\theta}{2})\right]$$

$\frac{\pi}{2}$

$$F(x) = \Psi(\underset{|v|^2}{A}) + \Psi(\underset{|v_*|^2}{B}) - \Psi(A\underset{\cos^2\frac{\theta}{2}}{x} + B\underset{\sin^2\frac{\theta}{2}}{(1-x)}) - \Psi(A(1-x) + Bx)$$

Based on the convexity, $F(x)$ takes the maximum at $x = \frac{1}{2}$, and is decreasing with respect to $|x - \frac{1}{2}|$

when $x = \frac{1}{2} \Leftrightarrow \cos^2\frac{\theta}{2} = \frac{1}{2} \Leftrightarrow \cos\frac{\theta}{2} = \frac{\sqrt{2}}{2} \Leftrightarrow \frac{\theta}{2} = \frac{\pi}{4} \Leftrightarrow \theta = \frac{\pi}{2}$

$$\ge 2\pi C_0 (\theta_2 - \theta_1)\left[\Psi(|v|^2) + \Psi(|v_*|^2) - \Psi(|v|^2\cos^2\tfrac{\theta_1}{2} + |v_*|^2\sin^2\tfrac{\theta_1}{2}) - \Psi(|v_*|^2\cos^2\tfrac{\theta_1}{2} + |v|^2\sin^2\tfrac{\theta_1}{2})\right]$$

there exist another constant C_2 independent of "n" such that

$$\ge \boxed{C_2 \left(\langle v\rangle^{2+k} 1_{\{\langle v\rangle \ge 2\langle v_*\rangle\}} + \langle v_*\rangle^{2+k} 1_{\{\langle v_*\rangle \ge 2\langle v\rangle\}}\right)}$$

where we need to take $x_1 = \cos^2\frac{\theta_1}{2}$ and $X = \frac{\langle v\rangle^2}{\langle v\rangle^2 + \langle v_*\rangle^2}$

$$x_1\Psi(|v|^2) + (1-x_1)\Psi(|v_*|^2) - \Psi(|v|^2 x_1 + |v_*|^2(1-x_1)) \qquad \Psi = (1+x)^{1+\frac{k}{2}}$$

$$= (\langle v\rangle^2 + \langle v_*\rangle^2)^{1+\frac{k}{2}} \left\{ x_1 X^{1+\frac{k}{2}} + (1-x_1)(1-X)^{1+\frac{k}{2}} - (x_1 X + (1-x_1)(1-X))^{1+\frac{k}{2}} \right\}$$

$\Updownarrow$

if $\langle v\rangle \ge 2\langle v_*\rangle$, $\frac{4}{5} \le X \le 1$

or if $\langle v_*\rangle \ge 2\langle v\rangle$, $0 \le X \le \frac{1}{5}$.

$$k_n(v,v_*) = -H_n(v,v_*) + G_n(v,v_*)$$

Since $\boxed{\int_{\mathbb{R}^3} \langle v\rangle^2 f_t^n(v)\,dv = \int_{\mathbb{R}^3} \langle v\rangle^2 f_0(v)\,dv}$

and we can select large enough $R>0$ (independent of n) such that $\int_{\mathbb{R}} f_t^n(v)\,dv > \frac{1}{2}$.

$$\Rightarrow \int_{\mathbb{R}^3} \langle v\rangle^{2+k} f_t^n(v)\,dv + C_2' \int_0^t \int_{\mathbb{R}^3} \langle v\rangle^{2+k} \underbrace{\min\{\langle v\rangle^\gamma, n^\gamma\}}_{\substack{|v-v_*|^\gamma \downarrow \\ \nearrow n\to\infty}} f_\tau^n(v)\,dv$$

$$\le \int_{\mathbb{R}^3} \langle v\rangle^{2+k} f_0(v)\,dv + C_3'\left(\int_{\mathbb{R}^3} \langle v\rangle^2 f_0(v)\,dv\right) \int_0^t \int_{\mathbb{R}^3} \langle v\rangle^{\max\{2,k\}} \min\{\langle v\rangle^\gamma, n^\gamma\} f_\tau^n(v)\,dv\,d\tau$$

for suitable R_0 s.t. $C_2' R_0^{\min\{2,k\}} \ge 2C_3' \int_{\mathbb{R}^3} \langle v\rangle^2 f_0(v)\,dv$.

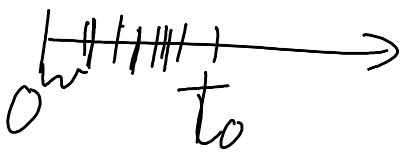
$$\Rightarrow C_2' \int_{|v|\ge R_0} \langle v\rangle^{2+k} \min\{\langle v\rangle^\gamma, n^\gamma\} f_\tau^n(v)\,dv \ge 2C_3'\left(\int_{\mathbb{R}^3} \langle v\rangle^2 f_0(v)\right) \int_{|v|\ge R_0} \langle v\rangle^{\max\{2,k\}} \min\{\langle v\rangle^\gamma, n^\gamma\} f_\tau^n(v)\,dv$$

$$\Rightarrow \sup_{t\in[0,T]} \int_{\mathbb{R}^3} \langle v\rangle^{2+k} f_t^n(v)\,dv + \int_0^T \int_{\mathbb{R}^3} \langle v\rangle^{2+k} \min\{\langle v\rangle^\gamma, n^\gamma\} f_\tau^n(v)\,dv\,d\tau \le C_T$$

\#

soft $\Leftrightarrow$ preserve

hard $\Leftrightarrow$ creat



t_0

for any $t_0>0$

$$\int \langle v\rangle^{2+k} f_0(v)\,dv < \infty,\ k>0 \Rightarrow \forall\, l>0,\ \sup_{t\ge t_0} \int_{\mathbb{R}^3} |v|^l f_t(v)\,dv < \infty$$

$\Downarrow$ relax

$\Psi(x) = \dots$ $\Downarrow$ $x=|v|^2$

$$\int \langle v\rangle^2 f_0(v)\,dv < \infty \ ? \Longrightarrow \int \langle v\rangle^2 \boxed{\rho(|v|^2)} f_0(v)\,dv < \infty$$

Lemma: (Mischler–Wennberg) $\qquad \overbrace{r\rho(r)}^{r=|v|^2}$

Let the initial datum $f_0(v)$ be $\int_{\mathbb{R}^3} f_0(v)\langle v\rangle^2 dv < \infty$.

$\boxed{\text{Then}}$ there exists a concave function $\rho(r)\in C^2$, depending on $f(v)$, such that $\rho(r)\to\infty$ as $r\to\infty$, $\boxed{\rho(r)r}$ is convex

$\boxed{r\rho'(r)\lesssim \log r, \qquad \rho'(r)\lesssim \frac{\log r}{r}}$

and such that for all $\varepsilon>0$ and $\alpha\in(0,1)$, $(\rho(r)-\rho(\alpha r))r^{\varepsilon}\to\infty$, as $r\to\infty$, and

$$\int_{\mathbb{R}^3}\langle v\rangle^2\rho(|v|^2)f_0(v)\,dv<\infty. \qquad // \int_{\mathbb{R}^3}\langle v\rangle^2\rho(|v|^2)\,dF_0(v)<\infty.$$

Pf: Without less of generality, $\boxed{\int_{\mathbb{R}^3}\langle v\rangle^2 f_0(v)\,dv=2}$

(Hints) Take $0=r_0<r_1<r_2\cdots<r_j<\cdots$ such that

$$\int_{|v|\geq r_j}\langle v\rangle^2 f_0(v)\leq 2^{-(j-1)}$$

One can further take $\{r_j\}_{j=1}^{\infty}$ such that $r_j>e^j$ such that $r_{j+1}-r_j>r_j-r_{j-1}$.

$\Rightarrow$ let $\boxed{\rho_1(r)}$ is linear in $[r_j, r_{j+1})$ such that $\rho_1(r_j)=j$.

$$\Rightarrow \int_{\mathbb{R}^3}\langle v\rangle^2\rho_1(|v|^2)f_0(v)\,dv\leq\sum_{j=1}^{\infty}2^{-(j-1)}(j+1)<\infty \qquad \text{Concave}$$

$[r_0,r_1]$, $[r_1,r_2]$ ....

$$\Longrightarrow \rho(r)=\frac{1}{r}\int_0^r\frac{1}{y}\int_0^y\left(\rho_1(z)+1-\log(e+z)^{-1}\right)dz\,dy \qquad \begin{matrix} r\rho(r) \\ \text{convex?}\end{matrix}$$

$\boxed{\text{Check}}$ that $r\rho(r)$ is convex, $\rho(r)\leq\rho_1(r)+1$.

$$\int_{\mathbb{R}^3}\langle v\rangle^2\rho(|v|^2)f(v)\,dv<\infty, \quad r\rho'(r)\lesssim\log r.$$
$$\Uparrow$$
$$\rho_1(e^j)\leq j.$$

\#

Proposition: For $0<\gamma<2$, let $f_0(v)$ satisfy

$b_n \Phi_n(|v-v_*|)$ $\left| \int_{\mathbb{R}^3} |v|^2 f_0(v)\,dv < \infty \right|$ ✓

$\boxed{f_t^n}, \boxed{f_t}$, Then, for any fixed $T>0$, there exists a $C_T>0$ independent of "n", such that,

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^3} \langle v\rangle^2 \rho(|v|^2) f_t^n\,dv + \int_0^T \int_{\mathbb{R}^3} \langle v\rangle^{2-\varepsilon} \min\{\langle v\rangle^\gamma, n^\gamma\} f_\tau^n(v)\,dv\,d\tau < C_T$$

for a sufficiently small $0<\varepsilon<\frac{1}{2}$.

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^3} \langle v\rangle^2 \rho(|v|^2) f_t\,dv + \int_0^T \int_{\mathbb{R}^3} \langle v\rangle^{2-\varepsilon+\gamma} f_\tau(v)\,dv\,d\tau < C_T$$

Pf: Since the initial datum $f_0(v) \in P_2(\mathbb{R}^3)$,
$\Downarrow$
$\{\int f_0(v)|v|^2 dv < \infty\}$

for $\rho(r)$ in Lemma above, we have

$$\int_{\mathbb{R}^3} \langle v\rangle^2 \rho(|v|^2) f_0(v)\,dv < \infty$$

$\Rightarrow$ for each fixed n and $T>0$, we find

$$\sup_{0\le t\le T} \int_{\mathbb{R}^3} \boxed{\langle v\rangle^2 \rho(|v|^2)} f_t^n\,dv < \infty.$$

compare

$\overset{||}{\Psi(|v|^2)}$ Convex

by noting that $\int_{\mathbb{R}^3} \rho(|v|^2) f_t^n(v)\,dv \le \int_{\mathbb{R}^3} \langle v\rangle^2 f_t^n(v)\,dv$

\+

by step I in last Theorem $= \int_{\mathbb{R}^3} \langle v\rangle^2 f_0(v)\,dv.$

considering $W_\delta(v) = \frac{\langle v\rangle^2}{\langle \delta v\rangle^2} \rho(|v|^2)$, $\delta>0$

And, since $\rho(r)$ itself is just concave function.

$$\rho(2r) \le \rho(2r) + \rho(0) \le 2\rho(r),$$

$$W_\delta(v') \le \left(\frac{1+|v|^2}{1+\delta^2|v|^2} + \frac{1+|v_*|^2}{1+\delta^2|v_*|^2}\right) \rho(|v|^2 + |v_*|^2)$$

$$\le 4\max\{W_\delta(v), W_\delta(v_*)\}.$$

Similar to $\Psi_k(|v|^2) = (1+|v|^2)^{1+\frac{k}{2}} = \langle v\rangle^{2+k}$
$\Updownarrow$
$\Psi_k(x) = (1+x)^{1+\frac{k}{2}}$

replace $\Psi_K(x)$ by $\Psi(x) = x\rho(x) \Rightarrow \Psi(|v'|^2) = |v'|^2\rho(|v'|^2)$

$$\Psi(|v'|^2) = (Y(\theta) + Z(\theta)\cos\phi)\,\rho(Y(\theta) + Z(\theta)\cos\phi)$$

$$\le (Y(\theta) + Z(\theta)\cos\phi)(\rho(Y(\theta)) + \rho'(Y(\theta))Z(\theta)\cos\phi)$$

Therefore, we have

$$\int_0^\pi \Psi(|v'|^2)\,d\phi \le \boxed{\pi\Psi(Y(\theta))} + \frac{\pi}{2}Z^2(\theta)\rho'(Y(\theta))$$

similarly, for $\Psi(|v_*'|^2)$, we have $\quad G_n(v,v_*)$

$$\int_0^\pi \Psi(|v_*'|^2)\,d\phi \le \boxed{\pi\Psi(Y(\pi-\theta))} + \frac{\pi}{2}Z^2(\theta)\rho'(Y(\pi-\theta))$$

For $G_n(v,v_*)$, $\quad |Z| \le Y \quad \boxed{K_n(v,v_*) = -H_n(v,v_*) + G_n(v,v_*)}$

$$\boxed{G_n(v,v_*)} = \int_0^\pi b_n(\cos\theta)\sin\theta\,\frac{\pi}{2}Z^2(\theta)\left[\rho'(Y(\theta)) + \rho'(Y(\pi-\theta))\right]d\theta$$

$|Z|^2 \le |v-v_*|^2|v_*|^2\sin^2\theta$

$\le |v|^2|v_*|^2$

$\le \frac{\log Y(\theta)}{Y(\theta)} \qquad \le \frac{\log(Y(\pi-\theta))}{Y(\pi-\theta)}$

$$\le C|v|^{2-2\varepsilon}|v_*|^{2-2\varepsilon}\int_0^\pi \left(\frac{|Z|}{Y}\right)^{2\varepsilon} \underbrace{b_n(\cos\theta)\sin^{2-2\varepsilon}\theta\sin\theta\,d\theta}_{<\infty}$$

$$\boxed{\le C_1|v|^{2-2\varepsilon}|v_*|^{2-2\varepsilon}}$$

For $H_n(v,v_*)$ $\quad$ convexity $\leftrightarrow x\rho(x)$ $\quad |v|^2\cos^2\frac{\theta}{2} + |v_*|^2\sin^2\frac{\theta}{2}$

$$\boxed{H_n(v,v_*)} = 2\pi\int_0^\pi b_n(\cos\theta)\sin\theta\left[\Psi(|v|^2) + \Psi(|v_*|^2) - \Psi(Y(\theta)) - \Psi(Y(\pi-\theta))\right]d\theta$$

Consider $\boxed{F(x)} = \Psi(A) + \Psi(B) - \Psi(Ax + B(1-x)) - \Psi(A(1-x) + Bx)$

$[\theta_1,\theta_2] \subset [0,\frac{\pi}{2}]$ $\quad$ take maximum $x = \frac{1}{2}$, $\downarrow$ in $|x - \frac{1}{2}|$

$$\ge 2\pi\int_{\theta_1}^{\theta_2} b_n(\cos\theta)\sin\theta\left[\Psi(|v|^2) + \Psi(|v_*|^2) - \Psi(Y(\theta)) - \Psi(Y(\pi-\theta))\right]d\theta$$

$$\ge 2\pi(\theta_2-\theta_1)C_0\Big[|v|^2\rho(|v|^2) + |v_*|^2\rho(|v_*|^2) - ((1-\tau)|v|^2 + \tau|v_*|^2)$$

$$\cdot\rho((1-\tau)|v|^2 + \tau|v_*|^2) - ((1-\tau)|v_*|^2 + \tau|v_*|^2)$$

$$\cdot\rho(|v|^2 + (1-\tau)|v_*|^2)$$

$\ge 2\pi(\theta_2-\theta_1)C_0[\ldots]$

$$\geq 2|v|^2 \ldots \left[|v_*|^2 \rho(|v_*|^2) - (|v|^2+|v_*|^2)\rho(a|v_*|^2) \right]$$

where $a = \max\left\{ \frac{1+3\tau}{4}, 1-\frac{3\tau}{4} \right\}$. $\rho \uparrow$ increasing

$$(1-\tau)|v|^2 + \tau|v_*|^2 \leq \frac{(1+3\tau)|v_*|^2}{4}.$$

Since
$$\begin{cases} |v|^2 \rho(|v_*|^2) \lesssim |v|^{2-2\varepsilon}|v_*|^{2-2\varepsilon}, & \text{if } |v_*| \geq 2|v|. \\ |v_*|^2 \rho(|v|^2) \lesssim |v_*|^{2-2\varepsilon}|v|^{2-2\varepsilon}, & \text{if } |v| \geq 2|v_*|. \end{cases}$$

$\overset{\exists C_2, C_3}{\Longrightarrow}$
$$\boxed{H_n(v,v_*) \geq -C_3|v|^{2-2\varepsilon}|v_*|^{2-2\varepsilon} + C_2\left(|v|^{2-\varepsilon} 1_{\{|v|\geq 2|v_*|\}} + |v_*|^{2-\varepsilon} 1_{\{|v_*|\geq 2|v|\}} \right)}$$

As Theorem above,

$$\frac{d}{dt}\int \underset{\substack{\Downarrow \\ |v|^2\rho(|v|^2)}}{\Psi(|v|^2)} f_t^n \, dv = \int_{\mathbb{R}^3\times\mathbb{R}^3} \overset{-H_n(v,v_*)+G_n(v,v_*)}{\overset{\Downarrow}{K_n(v,v_*)}} f_t^n(v) f_t^n(v_*)\, dv\, dv_*$$

$$\underbrace{|v-v_*|^\gamma \chi_{\{|v-v_*|\leq n\}}}$$

Gronwall
$$\Longrightarrow \sup_{t\in[0,T]} \int_{\mathbb{R}^3} \langle v\rangle^2 \rho(|v|^2) f_t^n(v)\, dv + \int_0^T \int_{\mathbb{R}^3} \langle v\rangle^{2-\varepsilon} \min\{\langle v\rangle^\gamma, n^\gamma\} f_t^n(v)\, dv \leq C_T.$$

$\Longrightarrow$ for $f_t(v)$, take the limit $n\to\infty$.