Lecture_5

Thursday, 10 March 2022 3:34 pm

About $\boxed{L^p}$-estimate for homogeneous Boltzman Equation

$\begin{cases} \text{Existence and uniqueness of weak formulation } / L^1_2\text{-cutoff} \\ \text{Moment estimate } \int \langle v\rangle^l f_t(v)\,dv \quad \|f\|_{L^1_l} \overset{?}{\Rightarrow} \|f\|_{L^p} \Downarrow \|f\|_{H^s}\,? \end{cases}$

$\theta^{-(d-2+2\nu)}$

$$\begin{cases} \frac{\partial f}{\partial t}(t,v) = Q(f,f)(t,v) \\ f(t=0,v) = f_0(v) \end{cases}$$

Assumption: ① $B(|v-v_*|,\sigma) = |v-v_*|^\gamma\, b(\cos\theta)$, $0\le\gamma\le 1$

$b(s)\in L^\infty_{loc}((-1,1))$, $\boxed{b(s)|_{s\to1^-} \sim \frac{1}{(1-s)^{\frac{d-2+2\nu}{2}}}, \ \nu>-2}$ $\quad \cos\theta \triangleq s$ $\quad = \left(\sin\frac{2\theta}{2}\right)^{\frac{d-2+2\nu}{2}}$

$\Updownarrow$

$\sin^{d-1}\theta\, b(\cos\theta)|_{\theta\to0} \sim K\theta^{-1-2\nu}$, $0<\nu<2$

② Consider $f_0\ge0$ with finite mass and energy,

$\underbrace{f_0(1+|v|^2)^{\frac{q}{2}}}_{f_0\langle v\rangle^q} \in L^p(\mathbb{R}^d) \iff \int_{\mathbb{R}^d} |f_0(v)|^p \langle v\rangle^{pq}\,dv < \infty$.

for some $\boxed{1<p<\infty}$ and $q\ge0$

Functional space: let $1<p<\infty$,

$L^p_q(\mathbb{R}^d) := \{ f:\mathbb{R}^d\to\mathbb{R},\ \|f\|_{L^p_q}<\infty\}$

with the norm $\|f\|^p_{L^p_q(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |f(v)|^p \underset{\underset{(1+|v|^2)^{\frac12}}{\parallel}}{\langle v\rangle}^{pq}\,dv$

Theorem (L^p estimate)

$0<\nu<2$

let $\underset{|v-v_*|^\gamma b(\cos\theta)}{B}$ be the collision kernel satisfy the assumptions above and the q such that: $\quad Q(f,f) \sim \Delta f$

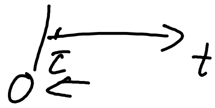
$$\begin{cases} \text{(i) } q \in \mathbb{R}_+, \text{ if } \nu > -1 \quad (\text{Remark: Cutoff case}) \\ \text{(ii) } pq > 2, \text{ if } \nu \in (-2,-1], \\ \text{(iii) } pq > 4 \text{ if } \nu \in (-3,-2]. \end{cases}$$

and f_0 be an initial datum $L^1_{\max(p,2)q+2} \cap L^p_q$

Then, ① there exists a weak solution f with B and f_0 such that $f \in L^\infty([0,\infty); L^p_q(\mathbb{R}^d))$

② if $\nu > 0$, $f \in L^\infty((\tau,\infty); L^p_r(\mathbb{R}^d))$ for all $\tau > 0$ and $r > q$.

Remark: It only works for hard potential

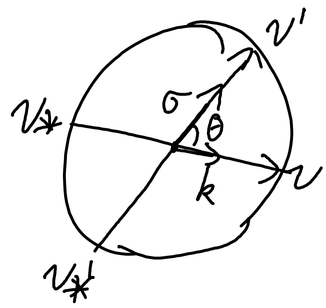


$$\frac{\partial f}{\partial f} = Q(f,f) \Rightarrow_{f^{p-1}\langle v\rangle^{pq}} \frac{\partial \int f \cdot f^{p-1} \langle v\rangle^{pq} dv}{\partial t} = \boxed{\int Q(f,f) f^{p-1} \langle v\rangle^{pq} dv}$$

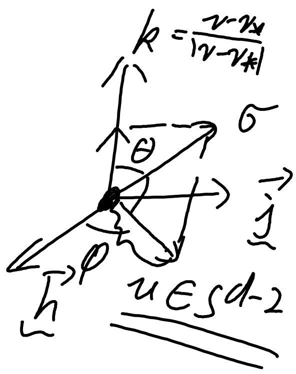
$$\Downarrow$$

$$\frac{\partial \|f\|_{L^{pq}}}{\partial t}$$

Preparation: $k = \frac{v - v_*}{|v - v_*|}$, $\sigma = \frac{v' - v_*'}{|v' - v_*'|}$



$\sigma = \cos\theta\, k + \sin\theta\, u$



$\int Q(f,f) f^{p-1} \langle v\rangle^{pq} dv$

$$Q(f,f) = \int_{\mathbb{R}^d} \int_{S^{d-1}} B(|v - v_*|, \sigma)[f(v')f(v_*') - f(v)f(v_*)]\, d\sigma\, dv_*$$

$$\Downarrow$$

$$[B(|v-v_*|, \cos\theta) + B(|v-v_*|, \cos(\pi-\theta))] \mathbb{1}_{\cos\theta \geq 0}$$

or $0 \leq \theta \leq \frac{\pi}{2}$.

$$\boxed{v'} = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma \qquad (1)$$

Lemma: For $F(v)$, $\qquad v \mapsto v'$

$$\int_{\mathbb{R}^d} \int_{S^{d-1}} B(|v-v_*|, \cos\theta) F(v')\, dv\, d\sigma = \int \int \frac{1}{\dots} d\dots$$

$\int_{\mathbb{R}^d}\int_{S^{d-1}} \cdots \qquad \int_{\mathbb{R}^d}\int_{S^{d-1}} \cos^d(\frac{\theta}{2}) B(\frac{|v-v_*|}{\cos\frac{\theta}{2}}, \cos\theta) F(v)\, dv\, d\sigma.$

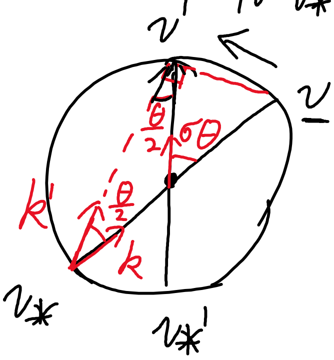
<u>Pf</u>: From (1),

$$v' = v_* + \frac{|v-v_*|}{2}(k + k\sigma)$$

For each σ and fixed v_*, we just perform change of variable $\boxed{v \mapsto v'}$, which is well-defined for $\boxed{\cos\theta > 0}$

$$\left|\frac{dv'}{dv}\right| = \left|\frac{1}{2}I + \frac{1}{2}k\otimes\sigma\right| = \frac{1}{2^d}(1 + k\cdot\sigma) = \frac{k'\cdot\sigma}{2^{d-1}}$$

where $k = \frac{v-v_*}{|v-v_*|}$, $k' = \frac{v'-v_*}{|v'-v_*|}$



$$k'\cdot\sigma = \cos\frac{\theta}{2} \geq \frac{1}{\sqrt{2}} \qquad 0 \leq \theta \leq \frac{\pi}{2}$$

similarly, $v' \mapsto v = \psi_\sigma(v')$

$$|v_* - \psi_\sigma(v')| \cos\frac{\theta}{2} = |v' - v_*|$$

$$\Rightarrow |v_* - \psi_\sigma(v')| = \frac{|v'-v_*|}{k'\cdot\sigma}$$

$$\Rightarrow |v_* - \psi_\sigma(v)| = \frac{|v-v_*|}{k\cdot\sigma}$$

$$\int_{\mathbb{R}^d}\int_{S^{d-1}} B(|v-v_*|, \cos\theta) F(v')\, dv\, d\sigma$$

$$\overset{v\mapsto v'}{=} \int_{\mathbb{R}^d}\int_{S^{d-1}} B(|v-v_*|, \cos\theta) F(v') \left|\frac{dv}{dv'}\right| dv'\, d\sigma$$

$$= \int_{k\cdot\sigma > \frac{1}{\sqrt{2}}} B(|\psi_\sigma(v)-v_*|, \cos\theta) F(v') \frac{2^{d-1}}{(k'\cdot\sigma)^2}\, dv'\, d\sigma$$

$$\overset{v'\to v}{=} \int_{\mathbb{R}^d}\int_{\substack{k\cdot\sigma > \frac{1}{\sqrt{2}} \\ 0\leq\theta\leq\frac{\pi}{4}}} \frac{2^{d-1}}{(k\cdot\sigma)^2} B(\frac{|v-v_*|}{k\cdot\sigma}, 2(k\cdot\sigma)^2 - 1)\, d\sigma\, F(v)\, dv$$

$$= |S^{d-2}| \int_{\mathbb{R}^d}\int_0^{\frac{\pi}{4}} \sin^{d-2}\theta \frac{2^{d-1}}{\cos^2\theta} B(\frac{|v-v_*|}{\cos\theta}, \cos 2\theta)\, d\theta\, F(v)\, dv$$

$$\overset{2\theta\to\theta}{=} |S^{d-1}| \int_{\mathbb{R}^d}\int_0^{\frac{\pi}{2}} \frac{\sin^{d-2}\theta}{\cos^d\frac{\theta}{2}} B(\frac{|v-v_*|}{\cos\frac{\theta}{2}}, \cos\theta)\, d\theta\, F(v)\, dv.$$

$$=\int_{\mathbb{R}^d}\int_{S^{d-1}} \frac{1}{\cos^d\frac{\theta}{2}} B\left(\frac{|v-v_*|}{\cos\frac{\theta}{2}}, \cos\theta\right) F(v)\, d\sigma\, dv.$$

<u>Proposition:</u> Let $\boxed{B}$ be a collision kernel satisfy assumption
$\underset{|v-v_*|^r b(\cos\theta)}{\shortparallel}$

given above. Then, for all $p>1$, $q\in\mathbb{R}$ and f, g non-negative we have:

$$\int_{\mathbb{R}^d} Q(g,f)\, f^{p-1}(v) \langle v\rangle^{pq} dv$$

$$\leq \int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{S^{d-1}} |v-v_*|^r b(\cos\theta)\left[\left(\cos\frac{\theta}{2}\right)^{-\frac{d+r}{p'}}-1\right]\langle v\rangle^{pq} f^p(v) g(v_*)\, d\sigma\, dv_* dv$$

$\boxed{\phi(v')-\phi(v)}$

$$+\int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{S^{d-1}} \frac{1}{p}\left(\cos\frac{\theta}{2}\right)^{-\frac{d+r}{p'}} |v-v_*|^r b(\cos\theta)\left[\langle v'\rangle^{pq}-\langle v\rangle^{pq}\right] f^p(v) g(v_*)\, d\sigma\, dv_* dv.$$

<u>Pf</u>: step 1: Apply the pre-post collisional change of variables $J=\left|\frac{\partial(v', v_*')}{\partial(v, v_*)}\right|=1$

$$\int_{\mathbb{R}^d} Q(g,f) f^{p-1}(v)\langle v\rangle^{pq} dv$$

$$=\int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{S^{d-1}} [g(v_*')f(v')-g(v_*)f(v)]\, f^{p-1}\langle v\rangle^{pq}|v-v_*|^r b(\cos\theta)\, d\sigma\, dv_* dv$$

$$=\int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{S^{d-1}} [\langle v'\rangle^{pq} f^{p-1}(v') f(v) g(v_*) - \langle v\rangle^{pq} f^{p-1}(v) f(v) g(v_*)]\, |v-v_*|^r b(\cos\theta)\, d\sigma\, dv_* dv$$

By taking the Young's inequality, for all $\mu=\boxed{\mu(\theta)}>0$

$$f^{p-1}(v') f(v) = \left(\frac{f(v')}{\mu^{\frac{1}{p}}}\right)^{p-1}\left(\mu^{1-\frac{1}{p}} f(v)\right) \leq \left(1-\frac{1}{p}\right)\mu^{-1} f^p(v') + \frac{1}{p}\mu^{p-1} f^p(v)$$

$$\leq \int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{S^{d-1}}\left[\left(1-\frac{1}{p}\right)\mu^{-1} f^p(v')\langle v'\rangle^{pq} + \frac{1}{p}\mu^{p-1} f^p(v)\langle v'\rangle - \langle v\rangle^{pq} f^p(v)\right] g(v_*)|v-v_*|^r b(\cos\theta)\, d\sigma\, dv_* dv.$$

use the $v\mapsto v'$

$$\le \int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{S^{d-1}} \Big[\Big(1-\frac{1}{p}\Big)\mu^{-1} f^p(v)\langle v\rangle^{pq}\Big(\cos\frac{\theta}{2}\Big)^{-d-\gamma} + \frac{1}{p}\mu^{p-1}\langle v'\rangle^{pq} f^p(v) - \langle v\rangle^{pq} f^p(v)\Big] g(v_*)\,|v-v_*|^\gamma b(\cos\theta)\,d\sigma\,dv_*\,dv.$$

$$= \int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{S^{d-1}} \langle v\rangle^{pq}|v-v_*|^\gamma b(\cos\theta) f^p(v) g(v_*)\Big[\Big(1-\frac{1}{p}\Big)\mu^{-1}\Big(\cos\frac{\theta}{2}\Big)^{-d-\gamma} + \frac{1}{p}\mu^{p-1} - 1\Big]d\sigma\,dv_*\,dv$$

$$+ \int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{S^{d-1}} \frac{1}{p}\mu^{p-1}\big[\langle v'\rangle^{pq} - \langle v\rangle^{pq}\big] f^p(v) g(v_*)|v-v_*|^\gamma b(\cos\theta)\,d\sigma\,dv_*\,dv.$$

step 2: By selecting $\mu(\theta) = \big(\cos\frac{\theta}{2}\big)^{-\frac{d+\gamma}{p}}$.

then, we can obtain

$$\begin{cases} \int_{\mathbb{R}^d} Q(g,f) f^{p-1}(v)\langle v\rangle^{pq}\,dv \\ \le \int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{S^{d-1}} |v-v_*|^\gamma b(\cos\theta)\Big[\Big(\cos\frac{\theta}{2}\Big)^{-\frac{d+\gamma}{p'}} - 1\Big]\langle v\rangle^{pq} f^p(v) g(v_*)\,d\sigma\,dv_*\,dv \\ + \int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{S^{d-1}} \frac{1}{p}\Big(\cos\frac{\theta}{2}\Big)^{-\frac{d+\gamma}{p'}}|v-v_*|^\gamma b(\cos\theta)\big[\langle v'\rangle^{pq} - \langle v\rangle^{pq}\big] f^p(v) g(v_*)\,d\sigma\,dv_*\,dv. \end{cases}$$

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Remark: If we select $\boxed{\mu(\theta) = \big(\cos\frac{\theta}{2}\big)^{-\frac{d+\gamma}{p}-q}}$

the following inequality holds:

$$\begin{cases} \int_{\mathbb{R}^d} Q(g,f)(v) f^{p-1}(v)\langle v\rangle^{pq}\,dv \\ \le \int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{S^{d-1}} \langle v\rangle^{pq}|v-v_*|^\gamma b(\cos\theta)\Big[\Big(\cos\frac{\theta}{2}\Big)^{\overset{>0}{\boxed{q-\frac{d+\gamma}{p'}}}} - 1\Big] f^p(v) g(v_*)\,d\sigma\,dv_*\,dv \\ + \int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{S^{d-1}} \frac{1}{p}\Big(\cos\frac{\theta}{2}\Big)^{-q(p-1)-\frac{d+\gamma}{p'}}|v-v_*|^\gamma b(\cos\theta)\Big[\langle v\rangle^{pq} - \Big(\cos\frac{\theta}{2}\Big)^{\boxed{pq}}\langle v\rangle^{pq}\Big] \\ \quad f^p(v) g(v_*)\,d\sigma\,dv_*\,dv. \end{cases}$$

Corollary: (bound for $\int_{\mathbb{R}^d} Q(g,f) f^{p-1}\langle v\rangle^{pq}\,dv$.)

Let B be a collision kernel satisfy Assumptions above.

f, g non-negative and $q \in \mathbb{R}$.

If $\boxed{pq \geq 2}$ for $\gamma \in (-2,-1]$ and $\boxed{pq \geq 4}$ for $\gamma \in (-3,-2]$.

then,

$$\int_{\mathbb{R}^d} Q(g,f) f^{p-1} \langle v \rangle^{pq} dv \leq C_{p,d,\gamma}(b) \|g\|_{L^1_{pq+\gamma}} \|f\|^p_{L^p_{q+\frac{\gamma}{p}}}$$

where $C_{p,d,\gamma}(b) = cst(p,d,\gamma)\Big(\underbrace{\int_{S^{d-1}} b(\cos\theta)(1-\cos\theta)\, d\sigma}_{\Downarrow\ O(\theta^2)}\Big)$

Pf: Thanks to the Proposition above,

we can split the integration into three parts:

$$\int_{\mathbb{R}^d} Q(g,f) f^{p-1} \langle v \rangle^{pq} dv \leq I_1 + I_2 + I_3$$

where,

$$I_1 := \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{S^{d-1}} b(\cos\theta) |v-v_*|^\gamma \overbrace{\left[\left(\cos\tfrac{\theta}{2}\right)^{-\frac{d+\gamma}{p'}} - 1\right]}^{O(\theta^2)} \langle v \rangle^{pq} f^p(v) g(v_*) d\sigma \, dv_* dv.$$

$$I_2 := \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{S^{d-1}} b(\cos\theta) |v-v_*|^\gamma \frac{1}{p} \left[\left(\cos\tfrac{\theta}{2}\right)^{-\frac{d+\gamma}{p'}} - 1\right] \left[\langle v' \rangle^{pq} - \langle v \rangle^{pq}\right] f^p(v) g(v_*) d\sigma \, dv_* dv.$$

$$I_3 := \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{S^{d-1}} b(\cos\theta) |v-v_*|^\gamma \frac{1}{p} \left[\langle v' \rangle^{pq} - \langle v \rangle^{pq}\right] f^p(v) g(v_*) d\sigma \, dv_* dv.$$

For I_1, I_2, we can use the relation:

$$\left[\left(\cos\tfrac{\theta}{2}\right)^{-\frac{d+\gamma}{p'}} - 1\right]\Big|_{\theta \to 0} \sim \frac{d+\gamma}{4p'} \underbrace{(1-\cos\theta)}_{O(\theta^2)}$$

For I_3, we'd better know estimate $\left| \int_{S^{d-1}} B \left(\langle v' \rangle^{pq} - \langle v \rangle^{pq}\right) d\sigma \right|$

$\langle v \rangle = (1+|v|^2)^{\frac{1}{2}}$

$k = \frac{v-v_*}{|v-v_*|}$

$u \in k^\perp$

Lemma: For all $\alpha \geq 1$. $\Leftrightarrow pq \geq 2$

$$\left| \int_{u \in S^{d-2}} \left[\langle v' \rangle^{2\alpha} - \langle v \rangle^{2\alpha}\right] du \right| \leq C_\alpha \left(\sin\tfrac{\theta}{2}\right) \langle v \rangle^{2\alpha} \langle v_* \rangle^{2\alpha} \sim O(\theta)$$

For all $\alpha \geq 2 \Leftarrow$ P9 24.

$$\left|\int_{u\in S^{d-2}} \left[\langle v'\rangle^{2\alpha} - \langle v\rangle^{2\alpha}\right] du\right| \leq C_\alpha \left(\sin^2\frac{\theta}{2}\right) \langle v\rangle^{2\alpha} \langle v_*\rangle^{2\alpha}$$

Pf: $|v'|^2 = Y(\theta) + \cancel{Z(\theta)\cos\phi}$

$$= |v|^2\cos^2\frac{\theta}{2} + |v_*|^2\sin^2\frac{\theta}{2} + 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}|v-v_*|u\cdot v_*$$

if we introduce another representation for $x \in [0, \frac{\sqrt{2}}{2}]$, ($x = \sin\frac{\theta}{2}$)

$$R_\alpha(x) = \int_{u\in S^{d-2}} \left[\left(1+|v|^2(1-x^2) + |v_*|^2x^2 + 2x\sqrt{1-x^2}\,|v-v_*|\,u\cdot v_*\right)^\alpha - (1+|v|^2)^\alpha\right] du.$$

$\Updownarrow$

$$R_\alpha\left(\sin\frac{\theta}{2}\right) = \int_{u\in S^{d-2}} \left[(1+|v'|^2)^\alpha - (1+|v|^2)^\alpha\right] du.$$

Considering $R_\alpha(x)$ is even and $R_\alpha(0) = 0$, we have

$$\begin{cases} R_\alpha(x) = x\int_0^1 R_\alpha'(sx)\,ds & \Rightarrow |R_\alpha(x)| \leq \frac{\sqrt{2}}{2}\int_0^1 |R_\alpha'(sx)|\,ds \\ R_\alpha(x) = x^2\int_0^1 (1-s)R_\alpha''(sx)\,ds. \end{cases}$$

We just need to calculate $R_\alpha'(x)$

$$\underset{\substack{\sin\frac{\theta}{2},\ 0<\theta<\frac{\pi}{2} \\ x\in[0,\frac{\sqrt{2}}{2}],\ \boxed{\alpha\geq 1}}}{R_\alpha'(x)} = \alpha\int_{u\in S^{d-2}} \Big[-2x|v|^2 + 2x|v_*|^2 + 2(1-x^2)^{\frac{1}{2}}|v-v_*|u\cdot v_* - 2x^2(1-x^2)^{-\frac{1}{2}}|v-v_*|u\cdot v_*\Big]\cdot\Big[1+|v|^2(1-x^2) + |v_*|^2x^2 + 2x\sqrt{1-x^2}\,|v-v_*|u\cdot v_*\Big]^{\alpha-1} du.$$

and

$$\underset{\boxed{\alpha\geq 2}}{R_\alpha''(x)} = \alpha(\alpha-1)\int_{u\in S^{d-2}} \Big[-2x|v|^2 + 2x|v_*|^2 + 2(1-x^2)^{\frac{1}{2}}|v-v_*|u\cdot v_* - 2x^2(1-x^2)^{-\frac{1}{2}}|v-v_*|u\cdot v_*\Big]^2\cdot\Big[1+|v|^2(1-x^2) + |v_*|^2x^2 + 2x\sqrt{1-x^2}\,|v-v_*|u\cdot v_*\Big]^{\alpha-2} du$$

$$+ \alpha\int_{u\in S^{d-2}} \Big[-2|v|^2 + 2|v_*|^2 - 2x(1-x^2)^{-\frac{1}{2}}|v-v_*|u\cdot v_* - 2|v-v_*|u\cdot v_*\left(2x(1-x^2)^{-\frac{1}{2}} + x^3(1-x^2)^{-\frac{3}{2}}\right)\Big]\Big[1+|v|^2(1-x^2) + |v_*|^2x^2 + 2x\sqrt{1-x^2}\,|v-v_*|u\cdot v_*\Big]^{\alpha-1}$$

du.

$\Rightarrow$ For $x \in [0, \frac{\sqrt{2}}{2}]$, $\alpha \geq 1$

$$|R_\alpha'(x)| \leq C_\alpha \langle v \rangle^{2\alpha} \langle v_* \rangle^{2\alpha}.$$

if $\underline{\alpha \geq 2}$

$$|R_\alpha''(x)| \leq C_\alpha' \langle v \rangle^{2\alpha} \langle v_* \rangle^{2\alpha} \quad \#$$

Let return back to I_3,

$$I_3 := \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{S^{d-1}} \frac{1}{p} |v - v_*|^\gamma b(\cos\theta) \underbrace{[\langle v' \rangle^{pq} - \langle v \rangle^{pq}]}_{\substack{\| \\ R_\alpha(\sin\frac{\theta}{2})}} f^p(v) g(v_*) \, d\sigma dv_* dv.$$

$$= \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_0^\pi \frac{1}{p} \underbrace{|v - v_*|^\gamma}_{\leq \langle v \rangle^\gamma \langle v_* \rangle^\gamma} b(\cos\theta) \boxed{\int_{u \in \mathbb{R}^{d-1}} [\langle v' \rangle^{pq} - \langle v \rangle^{pq}] \, du} \sin^{d-1}\theta \, d\theta \; f^p(v) g(v_*) \, dv_* dv$$

$\cos\theta k + \sin\theta u$

Lemma $\Rightarrow$ $2\alpha = pq$, $\sin^2\frac{\theta}{2} = \frac{1 - \cos\theta}{2}$

$$\leq \int_{\mathbb{R}^d \times \mathbb{R}^d} \left[\int_0^\pi b(\cos\theta) \sin^{d-1}\theta \left(\frac{1 - \cos\theta}{2} \right) d\theta \right] \langle v \rangle^{p(q + \frac{\gamma}{p})} \langle v_* \rangle^{pq + \gamma} f^p(v) g(v_*) \, dv_* dv$$

$$\int_{S^{d-1}} b(\cos\theta)(1 - \cos\theta) \, d\sigma$$

$$\leq C_{p,d,\gamma}(b) \|f\|_{L^p_{q + \frac{\gamma}{p}}}^p \|g\|_{L^1_{pq+\gamma}}$$

$\#$.

Proposition (bound $\int_{\mathbb{R}^d} Q(f, f) f^{p-1}(v) \langle v \rangle^{pq} dv$). cutoff version

Let B satisfy the assumption above.
Moreover, we suppose $b(\cos\theta)$ has its support in $[\theta_0, \frac{\pi}{2}]$
Then, for all $p > 1$, $q \geq 0$ and f non-negative with $\|f\|_{L^1_{pq+2}} < \infty$
we have

$$\int_{\mathbb{R}^d} Q(f, f)(v) f^{p-1}(v) \langle v \rangle^{pq} dv \leq C^+(b) \|f\|_{L^p_q}^p - K^-(b) \|f\|_{L^p_{q + \frac{\gamma}{p}}}^p.$$

with $C^+(b) = C^+(\int_{S^{d-1}} b\,d\sigma)$, $K^-(b) = K^-(\int_{S^{d-1}} b\,d\sigma)$

where C^+, K^- are strictly positive constants.

upper bound $\|f\|_{L^1_{pq+2}}$ lower bound $\|f\|_{L^1}$

C^+ depends on θ_b.

Pf: For $\int_{\mathbb{R}^d} Q(f,f) f^{p-1} \langle v\rangle^{pq} dv$

$$\le \int_{\mathbb{R}^d} Q^+(f,f) f^{p-1}(v)\langle v\rangle^{pq} dv - \int_{\mathbb{R}^d} Q^-(f,f) f^{p-1}(v)\langle v\rangle^{pq} dv.$$

For the loss term, $Q^-(f,f) = f\,\mathcal{L}(f)$

$$= f \int_{\mathbb{R}^d}\int_{S^{d-1}} |v-v_*|^{\gamma} b_c(\cos\theta) f(v_*) dv_*$$

$-Q^-(f,f) \le -K_0 \|b\|_{L^1} f\langle v\rangle^{\gamma}$

$0 \le \gamma \le 1$

$\ge \langle v\rangle^{\gamma} - C\langle v_*\rangle^{\gamma}$

$Q = Q^+ - Q^- + C_0\|b\|_{L^1} f$

$$\ge f\langle v\rangle^{\gamma} \|b\|_{L^1} \|f\|_{L^1}$$

K_0: lower bound of $\|f\|_{L^1}$

$-\int Q^-(f,f) f^{p-1}(v)\langle v\rangle^{pq} dv.$ $-f\|b\|_{L^1}\|f\|_{L^1_\gamma}$

C_0: upper bound $\|f\|_{L^1_\gamma}$

$$\le -K_0 \|b\|_{L^1(S^{d-1})} \|f\|^p_{L^p_{q+\frac{\gamma}{p}}} + C_0 \|b\|_{L^1(S^{d-1})} \|f\|^p_{L^p_q}$$

$K_0 > 0$ depend on lower bound on $\|f\|_{L^1}$

$C_0 > 0$ depends on upper bound $\|f\|_{L^1_\gamma}$.

On the other hand,

$$\int_{\mathbb{R}^d} Q^+(f,f) f^{p-1}(v)\langle v\rangle^{pq} dv = \int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{S^{d-1}} f_*' f' f^{p-1}\langle v\rangle^{pq} b_c(\cos\theta)|v-v_*|^{\gamma} d\sigma\, dv\, dv_*$$

dual argument $\boxed{1_{|v|\le r}} \cup \boxed{1_{|v|>r}}$

$\Longrightarrow$ split

$|v|\le r$: $I_1 = \int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{S^{d-1}} f^{p-1}\langle v\rangle^{pq} b_c(\cos\theta)|v-v_*|^{\gamma}(f\chi_r)' f_*'\, d\sigma\, dv_*\, dv$

$|v|>r$: $I_2 = \int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{S^{d-1}} f^{p-1}\langle v\rangle^{pq} b_c(\cos\theta)|v-v_*|^{\gamma}(f\chi_{r^c})' f_*'\, d\sigma\, dv_*\, dv.$

where $\chi_r(v) = 1_{|v|\le r}$

$\chi_{r^c}(v) = 1 - 1_{|v|<r}$.

Then, for I_1,

$$I_1 = \int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{S^{d-1}} f^{p-1}\langle v\rangle^{pq} b_c(\cos\theta)|v-v_*|^\gamma (f\chi_r)' f_*' \, d\sigma\, dv_*\, dv.$$

$$\overset{v,v_*\mapsto v',v_*'}{=} \int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{S^{d-1}} (f')^{p-1}\langle v'\rangle^{pq} b_c(\cos\theta)|v-v_*|^\gamma (f\chi_r) f_* \, d\sigma\, dv_*\, dv.$$

$\theta_0 \le \theta \le \frac{\pi}{2}$

$$\le \int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{S^{d-1}} f_* \left[(1-\tfrac{1}{p})\mu_1^{-1} f^p(v') + \tfrac{1}{p}\mu_1^{p-1}(f\chi_r)^p(v)\right]\langle v'\rangle^{pq}$$

$$b_c(\cos\theta)\underbrace{|v-v_*|^\gamma}_{\le \langle v\rangle^\gamma \langle v_*\rangle^\gamma,\ 0<\gamma<1} d\sigma\, dv_*\, dv.$$

$\boxed{v\mapsto v'}$

$$\le \|b\|_{L^1(S^{d-1})}\Big[(1-\tfrac{1}{p})\mu_1^{-1}\|f\|_{L^1_\gamma}(\cos\tfrac{\pi}{4})^{-d-\gamma}\|f\|^p_{L^p_{q+\frac{\gamma}{p}}}$$

$$+\tfrac{1}{p}\mu_1^{p-1}\|f\|_{L^1_{pq+\gamma}}\underbrace{\|f\chi_r\|^p_{L^p_{q+\frac{\gamma}{p}}}}_{\le \|f\|^p_{L^p_q}}.$$

As for I_2,

$$I_2 = \int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{S^{d-1}} \underset{\substack{\Downarrow\\ \sigma\mapsto-\sigma\\ \Downarrow\\ \text{support in } [\frac{\pi}{2},\pi-\theta_0]}}{\overset{[\theta_0,\frac{\pi}{2}]}{\tilde b_c(\cos\theta)}}|v-v_*|^\gamma f'(f\chi_{r^c})(v_*')f^{p-1}\langle v\rangle^{pq} d\sigma\, dv_*\, dv$$

$$\overset{(v,v_*)\mapsto(v',v_*')}{=} \int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{S^{d-1}} (f\chi_{r^c})(v_*)\underset{|v|>r}{f(v)}\langle v\rangle^{pq} f^{p-1}(v')\tilde b_c(\cos\theta)|v-v_*|^\gamma d\sigma\, dv_*\, dv$$

$$\le \int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{S^{d-1}} (f\chi_{r^c})(v_*)\Big[(1-\tfrac{1}{p})\mu_2^{-1} f^p(v') + \tfrac{1}{p}\mu_2^{p-1} f^p\Big]\langle v\rangle^{pq} \tilde b_c(\cos\theta)\underbrace{|v-v_*|^\gamma}_{\langle v\rangle^\gamma} d\sigma\, dv_*\, dv$$

$\boxed{v\mapsto v'}$

$$\le \|b\|_{L^1(S^{d-1})}\Big[(1-\tfrac{1}{p})(\sin\tfrac{\theta_0}{2})^{-d-\gamma}\mu_2^{-1}\|f\chi_{r^c}\|_{L^1_\gamma}\|f\|^p_{L^p_{q+\frac{\gamma}{p}}}$$

$\theta\in[\frac{\pi}{2},\pi-\theta_0]$

$$\underline{|J|} = (\cos\tfrac{\theta}{2})^{-\frac{d+\gamma}{p'}} \qquad +\tfrac{1}{p}\mu_2^{p-1}\|f\chi_{r^c}\|_{L^1_{pq+\gamma}}\|f\|^p_{L^p_{q+\frac{\gamma}{p}}}\Big]$$

$\boxed{\theta\in[\frac{\pi}{4},\frac{\pi}{2}-\theta_0]}$

thus,

$$I_2 \le \|b\|_{L^1(S^{d-1})}\Big[(1-\tfrac{1}{p})\mu_2^{-1}(\sin\tfrac{\theta_0}{2})^{-d-\gamma}(1+r^2)^{\frac{\gamma-2}{2}}\|f\|_{L^1_2}\|f\|^p_{L^p_{q+\frac{\gamma}{p}}} + \frac{1}{p}\mu_2^{p-1}\|f\|_{L^1_{pq+\gamma}}\|f\|^p_{L^p_{q+\frac{\gamma}{p}}}\Big].$$

Combining I_1 and I_2, we obtain the estimate for gain part:

$$\int_{\mathbb{R}^d} Q^+(f,f)(v) f^{p-1}(v)\langle v\rangle^{pq}dv$$

$$\le \|b\|_{L^1}\Big[\underbrace{\tfrac{1}{p}\mu_1^{p-1}(1+r^2)^{\frac{\gamma}{2}}\|f\|_{L^1_{pq+\gamma}}}_{C_1}\Big]\|f\|^p_{L^p_q} \leftarrow I_1$$

$$+\|b\|_{L^1}\Big[\underbrace{(1-\tfrac{1}{p})\mu_1^{-1}\cos(\tfrac{\pi}{4})^{-d-\gamma}}_{\substack{\uparrow \text{big} \\ I_1}} + \underbrace{(1-\tfrac{1}{p})\boxed{\mu_2^{-1}}(\sin\tfrac{\theta_0}{2})^{-d-\gamma}(1+r^2)^{\frac{\gamma-2}{2}}}_{\substack{\text{small} \Downarrow \quad \uparrow I_2 \quad \text{big}}}$$

$$+\frac{1}{p}\mu_2^{p-1}\Big]\boxed{\|f\|_{L^1_{pq+\gamma}}\|f\|^p_{L^p_{q+\frac{\gamma}{p}}}}$$

$\uparrow$ small

I_2.

$\gamma-2<0$

For θ_0 in the cutoff assumption fixed, we can select μ_2 ($\mu(\theta)$) small enough, then r big enought, then μ_1 big enough, such that:

$$\Big[(1-\tfrac{1}{p})\mu_1^{-1}(\cos\tfrac{\pi}{4})^{-d-\gamma} + (1-\tfrac{1}{p})\mu_2^{-1}(\sin\tfrac{\theta_0}{2})^{-d-\gamma}\langle r\rangle^{\gamma-2} + \tfrac{1}{p}\mu_2^{p-1}\Big]\|f\|_{L^1_{pq+\gamma}} \le \frac{K_0}{2}.$$

$$\begin{cases} -\int Q^-(f,f)f^{p-1}\langle v\rangle^{pq}dv \le -K_0\|b\|_{L^1}\|f\|^p_{L^p_{q+\frac{\gamma}{p}}} + C_0\|b\|_{L^1}\|f\|^p_{L^p_q} & (1) \\ \int Q^+(f,f)f^{p-1}\langle v\rangle^{pq}dv \le C_1\|b\|_{L^1}\|f\|^p_{L^p_q} + \frac{K_0}{2}\|b\|_{L^1}\|f\|^p_{L^p_{q+\frac{\gamma}{p}}} & (2) \end{cases}$$

(1) + (2)

$$\Rightarrow \boxed{\int Q(f,f)f^{p-1}\langle v\rangle^{pq}dv \le \underset{\substack{\Downarrow \\ \mu_1^{p-1}\|f\|_{L^1_{pq+\gamma}}\|b\|_{L^1}}}{C^+(b)}\|f\|^p_{L^p_q} - \underset{\substack{\Downarrow \\ K_0\|b\|_{L^1}}}{K^-(b)}\|f\|^p_{L^p_{q+\frac{\gamma}{p}}}.}$$

Cutoff

~~lower $\|f\|_{L^1}$~~ #.

Proposition 3: Let B satisfy the Assumption above, $p\in(1,+\infty)$ and $q\geq 0$. Suppose moreover $pq\geq 2$ if $\nu\in(-2,1]$ and $pq\geq 4$ if $\nu\in(-3,-2]$. Then, for $\boxed{f}$ non-negative with $\|f\|_{L^1_{pq+2}}<\infty$, we have

$$\int_{\mathbb{R}^d} Q(f,f)f^{p-1}(v)\langle v\rangle^{pq}dv \leq C^+\|f\|^p_{L^p_q} - K^-\|f\|^p_{L^p_{q+\frac{\gamma}{p}}}$$

for some positive constants C^+, K^-, depending on an upper bound on $\|f\|_{L^1_{pq+2}}$ and on a lower bound on $\|f\|_{L^1}$.

Pf: "Corollary" + "Proposition 2"

$$\begin{array}{ccc} \text{Corollary} & & \text{Proposition 2} \\ \Downarrow & & \Downarrow \\ \text{Non-Cutoff} & & \text{Cutoff} \\ \text{Rough} & & \Downarrow \\ \leq A\|f\|_X & & \leq C\|f\|_{X'} - K\|f\|_X \end{array}$$

By splitting of angular collision kernel $b(\cos\theta)$:

$$b(\cos\theta) = b_c^{\theta_0} + b_r^{\theta_0}, \quad \boxed{\theta_0} \text{ small enough.}$$

$$b_c^{\theta_0} = b(\cos\theta)\chi_{[\theta_0,\frac{\pi}{2}]}, \qquad b_r^{\theta_0} = b(\cos\theta)\chi_{[0,\theta_0]}$$

Under the splitting of "b", we can do the decomposition of $Q = Q_c + Q_r$

For the Q_r, we can apply Corollary

$$\int_{\mathbb{R}^d} Q_r(f,f)f^{p-1}\langle v\rangle^{pq}dv \leq \underbrace{C(b)\|f\|_{L^1_{pq+\gamma}}}_{\leq \|f\|_{L^1_{pq+2}}}\|f\|^p_{L^p_{q+\frac{\gamma}{p}}}$$

$$\leq A\|f\|^p_{L^p_{q+\frac{\gamma}{p}}} \qquad (3)$$

A

$$\boxed{C_{p,d,\gamma,\|f\|^{1}_{L^1_{pq+2}}} \underbrace{\int_{S^{d-1}} b(\cos\theta)\mathbb{1}_{[0,\theta_0]}(1-\cos\theta)d\sigma}_{<\infty}}$$

$\theta_0 \downarrow$

For Q_c, we can apply the Proposition 2,

$$\int_{\mathbb{R}^d} Q_c(f,f)f^{p-1}\langle v\rangle^{pq}dv \le C^+(b)\|f\|^p_{L^p_q} - K^-(b)\|f\|^p_{L^p_{q+\frac{\gamma}{p}}} \quad (4)$$

Hence, by selecting θ_0 small enough, such that $A < \frac{K^-(b)}{2}$

(3)+(4)

$$\Rightarrow \int_{\mathbb{R}^d}(Q_c+Q_r)(f,f)f^{p-1}\langle v\rangle^{pq}dv.$$

$$\le C^+\|f\|^p_{L^p_q} - \underbrace{K^-}_{:=\frac{K^-(b)}{2}}\|f\|^p_{L^p_{q+\frac{\gamma}{p}}} \qquad \#.$$

depends on $\begin{cases}\text{upper bound } \|f\|_{L^1_{pq+2}} \\ \text{lower bound } \|f\|_{L^1}\end{cases}$

Theorem: B satisfy assumption above

$$\begin{cases}(i)\ q\in\mathbb{R}_+, \text{ if } \gamma>-1 \\ (ii)\ pq>2, \text{ if } \gamma\in(-2,-1] \\ (iii)\ pq>4, \text{ if } \gamma\in(-3,-2]\end{cases}$$

(I)

$$f_0 \in L^1_{\max(p,2)q+2}\cap L^p_q \Rightarrow f(t,\cdot)\in L^p_q$$

Pf: $$\boxed{\frac{d\int_{\mathbb{R}^d} f\, f^{p-1}\langle v\rangle^{pq}dv}{dt} = p\int_{\mathbb{R}^3} Q(f,f)(v)f^{p-1}(v)\langle v\rangle^{pq}dv.} \quad (*)$$

$$f\ge 0 \xRightarrow[\text{conservation}]{\text{mass}} \|f(t,\cdot)\|_{L^1} = \|f_0(\cdot)\|_{L^1}.$$

$$f_0\in L^p_q \Rightarrow \|f_0\|_{L\log L}<\infty \Rightarrow \|f(t,\cdot)\|_{L\log L}<\infty.$$

$$f_0\in L^1_{pq+2} \xRightarrow[\text{moment}]{} \|f(t,\cdot)\|_{L^1_{pq+2}}<\infty \xRightarrow[\text{moment}]{} \|f\|_{L^1_k}<\infty,\ k<0$$

preservation $\qquad$ creation. for hard potential

$$f_0 \in L^p_q \overset{?}{\Longrightarrow} \|f(t,\cdot)\|_{L^p_q} < \infty$$

$\boxed{\text{From } (*) + \text{Proposition 3}} \Rightarrow \dfrac{d\|f\|^p_{L^p_q}}{dt} \le C\|f\|^p_{L^p_q} - k\|f\|^p_{L^p_{q+\frac{\gamma}{p}}}$

$$\Rightarrow \frac{d\|f\|^p_{L^p_q}}{dt} \le C\|f\|^p_{L^p_q}$$

$$\Rightarrow \|f\|^p_{L^p_q} \le e^{ct}\|f_0(\cdot)\|^p_{L^p_q}$$

$\Rightarrow \|f\|_{L^p_q}$ remain bounded (on all intervals $[0,T]$ for $T>0$) if it is initially finite.

(II) If $\gamma > 0$, then $f \in L^\infty((\tau,+\infty); L^p_r(\mathbb{R}^d))$ for all $\tau > 0$, and $r > q$, for $f_0(v) \in L^1_{\max(p,2)q+2} \cap L^p_q$

when $\tau \to 0^+$ blow up

Pf: For higher moments r in L^p norm,

By Hölder's inequality, we have

$$\|f\|_{L^p_r} \le \|f\|^\theta_{L^p_{q_1}} \|f\|^{1-\theta}_{L^p_{q_2}}$$

where $r = \theta q_1 + (1-\theta) q_2$.

Then, if we select $q_2 = 0$ and $q_1 = r + \frac{\gamma}{p}$, we obtain,

$$\|f\|_{L^p_r} \le \|f\|^{\frac{r}{r+\frac{\gamma}{p}}}_{L^p_{r+\frac{\gamma}{p}}} \|f\|^{\frac{\frac{\gamma}{p}}{r+\frac{\gamma}{p}}}_{L^p}$$

$$\Rightarrow \boxed{\|f\|_{L^p_{r+\frac{\gamma}{p}}} \ge \underset{\sup_{t\in[0,T]} \|f(t,\cdot)\|^{1+\frac{\gamma}{pr}}_{L^p}}{\underbrace{k_T}} \|f\|^{\frac{r+\frac{\gamma}{p}}{r}}_{L^p_r} := k_T \|f\|^{1+\frac{\gamma}{pr}}_{L^p_r}} \quad (6)$$

In $(*)$, we replay q by r

$$\frac{d\int f f^{p-1} \langle v\rangle^{pr} dv}{dt} = r\int Q(f,f) f^{p-1} \langle v\rangle^{pr} dv$$

$\Rightarrow d\|f\|^p_{L^p_r}$ ...

$$\frac{d}{dt} \leq -K\|f\|^{p}_{L^p_{r+\frac{\gamma}{p}}} + C\|f\|^p_{L^p_r}$$

$$\overset{\text{by (6)}}{\leq} -K_T\left(\|f\|^p_{L^p_r}\right)^{1+\frac{\gamma}{pr}} + C\|f\|^p_{L^p_r}$$

let $y = \|f\|^p_{L^p_r}$ $\Longrightarrow$ $\frac{dy}{dt} \leq -K_T\, y^{1+\frac{\gamma}{pr}} + Cy$ $\boxed{\text{Exercise}}$

$\xrightarrow[\text{ODE}]{\text{Bernoulli's}}$ $\|f(t,\cdot)\|_{L^p_r} < \infty$, for $t \in (0,T]$

more precisely, $\|f(t,\cdot)\|_{L^p_r} \leq \left[\frac{C}{K_T(1-e^{-C\frac{\gamma}{pr}t})}\right]^{\frac{r}{\gamma}}$ (7)

Remark: The upper bound given above in (7) cannot be optimal.

(III) Behavior for large times.

$\|f_0\|_{L^p_q} < \infty \underset{\text{uniformly}}{\Longrightarrow} \|f(t,\cdot)\|_{L^p_q} < \infty$ on $[0,T]$

If we want to get the uniform estimate, we have modification for $\underset{\text{cutoff version}}{\text{Proposition 2}}$ and then $\underset{\text{complete version}}{\text{Proposition 3}}$.

$\Downarrow$ stronger version.

Proposition 2*

B satisfy cutoff, i.e., $b(\cos\theta)$ has support in $[\theta_0, \frac{\pi}{2}]$.

for all $p>1$, $q\geq 0$, f non-negative $\boxed{\text{with bounded entropy and } L^1_{2q+2} \text{ norm}}$

$$\int_{\mathbb{R}^d} Q(f,f) f^{p-1} \langle v\rangle^{pq}\, dv \leq C^+(b)\|f\|^{p(1-\varepsilon)}_{L^p_q} - K^-(b)\|f\|^p_{L^p_{q+\frac{\gamma}{p}}}$$

$\varepsilon \in [0,1)$ is constant depends on d and p.

upper bound $\|f\|_{L\log L}$ and $\|f\|_{L^1_{2q+2}}$

lower bound $\|f\|_{L^1}$

Proposition 3*

B satisfy ascuptions. $p \in (1, +\infty)$ and $q \geq 0$, $\begin{cases} pq \geq 2, & \text{if } \gamma \in (-2, -1] \\ pq \geq 4, & \text{if } \gamma \in (-3, -2]. \end{cases}$

Then, for f non-negative with bounded entropy and $L^1_{\max\{pq, 2q\}+2}$ norm

$$\int_{\mathbb{R}^d} Q(f,f)(v) f^{p-1}(v) \langle v \rangle^{pq} \, dv \leq C^+ \|f\|_{L^p_q}^{p(1-\varepsilon)} - k^- \|f\|^p_{L^p_{q+\frac{\gamma}{p}}}.$$

Pf: Cutoff + Non-cutoff

$\Uparrow$ Proposition 2* $\quad$ $\Uparrow$ Corollary. #

$$\frac{d\|f\|^p_{L^p_q}}{dt} \overset{\text{Proposition 3*}}{\leq} C\|f\|^{p(1-\varepsilon)}_{L^p_q} - k\|f\|^p_{L^p_q}$$

$$\Leftrightarrow \frac{dy}{dt} \leq C y^{1-\varepsilon} - k y$$

$$\Rightarrow y(t) \leq \max\left\{ y(\tau); \left(\frac{C}{k}\right)^{\frac{1}{\varepsilon}} \right\}, \quad \forall t \geq \tau. \quad \#$$