Lecture_6

Thursday, 31 March 2022 2:34 pm

Measure - valued solution to the homogeneous Boltzmann.

$\Downarrow$

(Fourier transform)

$$\begin{cases} \frac{\partial f}{\partial t} = Q(f,f) \\ f(t=0,v) = F_0(v) \end{cases}, \quad f \text{ is probability density function}$$

$f(t,v)$, $t \in \mathbb{R}^1$, $v \in \mathbb{R}^3$

$$dF(v) = f(v)\,dv$$

1、Probabitity Measure:

(1.1) From the classical Probability perspective

For a measure space $(\Omega, B(\Omega), \mu)$

(i) $B(\Omega) \subset 2^{\Omega}$ is σ-algebra in Ω, $B(\Omega)$ is collection of subsets of Ω such that

$$\begin{cases} \bullet\ \phi \in B \\ \bullet\ A \in B \Rightarrow A^c \in B \\ \bullet\ \bigcup_{n=1}^{\infty} A_n \in B \text{ whenever } A_n \in B, \forall n. \end{cases}$$

(ii) μ is called measure, $\mu: B \longmapsto [0,\infty]$, satisfying:

$$\begin{cases} \bullet\ \mu(\phi) = 0 \\ \bullet\ \mu(\bigcup_{n=1}^{\infty} A_n) = \bigcup_{n=1}^{\infty} \mu(A_n) \text{ whenever } \{A_n\} \text{ is a disjoint countable family of the member of } B. \end{cases}$$

(iii) Ω is σ-finite, i.e. there exists a countable family $\{\Omega\}_n$ in B such that $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$ and $\mu(\Omega_n) < \infty$, $\forall n$.

The Borel measure "μ": μ is defined on every Borel set.

$\Downarrow$

generated by

The Radon measure "μ" is Borel measure ...

The Radon measure μ is Borel measure, ... all bounded open set.

- Locally finite: $\mu(K)<\infty$, for any compact set $K\subset\Omega$.
- Borel regular: For each $A\subset\Omega$, there exists a Borel set B such that $A\subset B$ and $\mu(A)=\mu(B)$.

The probability measure "$\underline{\mu}$": μ is Radon measure.

$$\mu(\Omega):=\int_\Omega d\mu(x)=1.$$

From the duality perspective:

$(L^1)^*=L^\infty$

$(L^\infty)^*\supset L^1$

- If $\Omega\subset\mathbb{R}^3$ is a bounded domain, then, the space of all Radon measures $\mathcal{M}(\bar\Omega)$ is defined as the dual space of $C(\bar\Omega)=C_c(\Omega)$ including all the continuous functions on $C(\bar\Omega)$.

Furthermore, we have $L^1(\Omega)\hookrightarrow\mathcal{M}(\bar\Omega)$, since for any $f\in L^1(\Omega)$

$$\mu(\phi):=\int_\Omega f(v)\phi(v)dv,\ \text{for any } \phi\in C(\bar\Omega).$$

hence, $\mu\in\mathcal{M}(\bar\Omega)$ with $\|\mu\|_{\mathcal{M}(\bar\Omega)}\le\|f\|_{L^1(\Omega)}$

$\Rightarrow$ the space of Radon measures represents a natural extension of the space of integrable function.

- If $\Omega=\mathbb{R}^3$: $C_0(\mathbb{R}^3)=\{\phi(v)\in C(\mathbb{R}^3);\ \lim_{|v|\to\infty}\phi(v)=0\}$.

then, the space of Radon measure $\mathcal{M}(\mathbb{R}^3)$ is defined as:

$$\mathcal{M}(\mathbb{R}^3):=\{\underline{\mu}:C_0\mapsto\mathbb{R},\ \mu \text{ is linear functional s.t. } \exists C>0,\ |\mu(\phi)|\le C\|\phi\|_\infty.\ \forall\phi\in C_c\}$$

associated with the measure norm:

$$\|\mu\|_{\mathcal{M}(\mathbb{R}^3)}:=\sup_{\phi\in C_c(\mathbb{R}^3),\|\phi\|_\infty\le1}|\mu(\phi)|=\sup\left\{\int_{\mathbb{R}^3}\phi(v)d\mu(v);\ \phi\in C_c,\ \|\phi\|_\infty\le1\right\}$$

Remark: (i) it also holds for $\phi\in C_c^\infty(\mathbb{R}^3)=\mathcal{D}(\mathbb{R}^3)$ $\quad C_0(\mathbb{R}^3)=\overline{\mathcal{D}(\mathbb{R}^3)}^{\|\cdot\|_\infty}$

(ii) Thanks to the Riesz Representation Theorem:

$\mu\in\mathcal{M}(\mathbb{R}^3)$ linear functional $\xleftrightarrow{\text{one-to-one}}$ $\tilde\mu\in M(\mathbb{R}^3)$. measure $\tilde\mu:\mathcal{B}\mapsto[0,\infty)$

$$\mu(\phi) := \int_{\mathbb{R}^3} \phi(v)\, d\tilde{\mu}(v), \ \forall \phi \in C_0$$
$$\parallel$$
$$\langle \mu, \phi \rangle$$

Probability measure: $P(\mathbb{R}^3) := \{ \mu \in M^+(\mathbb{R}^3) \text{ with } \|\mu\|_{M(\mathbb{R}^3)} = 1 \}$

Definition. (Measure-valued solution homogeneous Boltzmann)

$F_t(v) \in C([0,\infty), P_\alpha(\mathbb{R}^3))$, $0 < \alpha \le 2$

$$:= \Big\{ F(v) \in P(\mathbb{R}^3) : \int_{\mathbb{R}^3} dF(v) = 1.$$
$$\text{if } 1 < \alpha \le 2: \int_{\mathbb{R}^3} v_j\, dF(v) = 0, \ j = 1,2,3.$$
$$\int_{\mathbb{R}} |v|^\alpha dF(v) < \infty \Big\}.$$

let $b(\cdot)$ satisfy cutoff or Non-cutoff assumption.

For any $F_0(v) \in P_\alpha(\mathbb{R}^3)$ with $0 < \alpha \le 2$.

We define $F_t(v) \in C([0,\infty), P_\alpha(\mathbb{R}^3))$ is a measure-valued solution, if it satisfies:

(1) For every $\boxed{\phi(v) \in C_b^2(\mathbb{R}^3)}$ and $t > 0$,

$$\int_0^t \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{S^2} b\Big(\frac{v - v_*}{|v - v_*|} \cdot \sigma\Big) \big|\phi(v_*') + \phi(v') - \phi(v_*) - \phi(v)\big|\, d\sigma\, dF_\tau(v)\, dF_\tau(v_*)\, d\tau$$

is finite.

(2) For every $\boxed{\phi(v) \in C_b^2(\mathbb{R}^3)}$

$$\int_{\mathbb{R}^3} \phi(v)\, dF_t(v) = \int_{\mathbb{R}^3} \phi(v)\, dF_0(v) + \frac{1}{2} \int_0^t \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{S^2} b\Big(\frac{v - v_*}{|v - v_*|} \cdot \sigma\Big)$$
$$\times [\phi(v_*') + \phi(v') - \phi(v_*) - \phi(v)]\, d\sigma\, dF_\tau(v)\, dF_\tau(v_*)\, d\tau.$$

(3) If $\alpha \ge 1$, then the momentum conservation law holds:

$$\forall t \ge 0, \int_{\mathbb{R}^3} v_j\, dF_t(v) = \int_{\mathbb{R}^3} v_j\, dF_0(v), \ j = 1,2,3.$$

$1, v_j, |v|^2$

$1 \in C_b^2(\mathbb{R}^3)$

mass conserved

(4) If $\alpha = 2$, then the energy conservation law holds:

$$\forall t \ge 0, \int_{\mathbb{R}^3} |v|^2 dF_t(v) = \int_{\mathbb{R}^3} |v|^2 dF_0(v).$$

$v_j, |v|^2 \notin C_b^2(\mathbb{R}^3)$

Remark: The continuity in time, the map $t\in[0,\infty)\longmapsto F_t(v)\in P_2$ is defined in the sense that

$$\lim_{t\to t_0}\int_{\mathbb{R}^3}\phi(v)\,dF_t(v)=\int_{\mathbb{R}^3}\phi(v)\,dF_{t_0}(v),\ \forall\phi\in C_0(\mathbb{R}^3)$$

with $C_\phi(\mathbb{R}^3):=\{\phi\in C(\mathbb{R}^3);\ \sup_{v\in\mathbb{R}^3}\frac{|\phi(v)|}{\langle v\rangle^2}<\infty,$

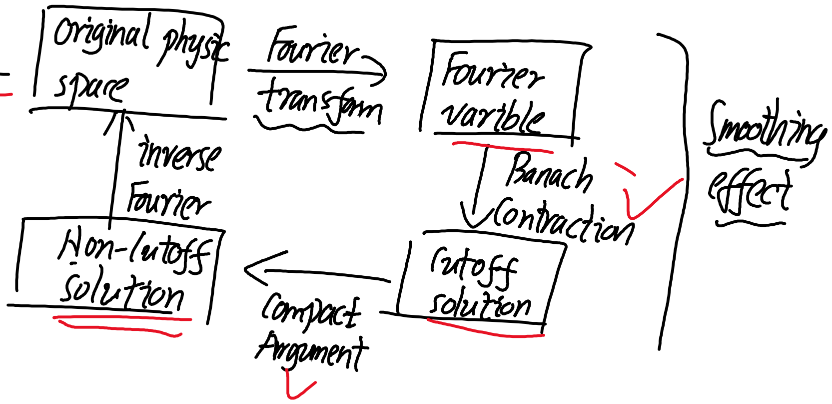
$$\langle v\rangle=(1+|v|^2)^{\frac{1}{2}}.\}.$$

we also need to define operator $L_b[\phi](v,v_*)$, for $\phi\in C_b^2(\mathbb{R}^3)$

$$\boxed{L_b[\phi](v,v_*):=\int_{S^2}b\left(\frac{v-v_*}{|v-v_*|}\cdot\sigma\right)[\phi(v_*')+\phi(v')-\phi(v_*)-\phi(v)]\,d\sigma}$$

$$\Rightarrow\int_{\mathbb{R}^3}\phi(v)\,dF_t(v)=\int_{\mathbb{R}^3}\phi(v)\,dF_0(v)+\frac{1}{2}\int_0^t\int_{\mathbb{R}^3\times\mathbb{R}^3}L_b[\phi]\,dF_\tau(v)\,dF_\tau(v_*)\,d\tau.$$

Fourier transform:



Fourier transformation to probability measure: in sense of Radon-Nikodym derivative.

For any given probability measure $F_t(v)$ or its density function $f(t,v)$

we define the corresponding characteristic function $\phi(\xi)$

$$\phi(\xi)=\hat{f}(\xi):=\int_{\mathbb{R}^3}e^{-iv\cdot\xi}f(v)\,dv=\int_{\mathbb{R}^3}e^{-iv\cdot\xi}\,dF(v)$$

how about $\mathcal{F}[Q(f,f)]$?

Proposition: Consider the $Q(g,f)$ with its collision kernel B being the Maxwellian molecule b.

$$Q(g,f)(v)=\int_{\mathbb{R}^3}\int_{S^2}b\left(\frac{v-v_*}{|v-v_*|}\cdot\sigma\right)[g(v_*')f(v')-g(v_*)f(v)]\,d\sigma\,dv_*$$

Then, the following formula holds: $\int g(v_*)e^{i\square\cdot v_*}dv_*$

$$\begin{cases} \mathcal{F}[Q^+(g,f)](\xi)=\int_{S^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right) \hat{g}(\xi^-)\hat{f}(\xi^+)\,d\sigma \\ \mathcal{F}[Q^-(g,f)](\xi)=\int_{S^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right) \hat{g}(0)\hat{f}(\xi)\,d\sigma. \end{cases}$$

$\int f(v) e^{-i\xi\cdot v}\,dv$

where $\xi^+=\frac{\xi}{2}+\frac{|\xi|}{2}\sigma$, $\xi^-=\frac{\xi}{2}-\frac{|\xi|}{2}\sigma$.

Pf: start with the weak form, for any test function ϕ:

$$\int_{\mathbb{R}^3} Q^+(g,f)(v)\phi(v)=\int_{\mathbb{R}^3\times\mathbb{R}^3}\int_{S^2} b\left(\frac{v-v_*}{|v-v_*|}\cdot\sigma\right)g(v_*)f(v)\,\phi(\underbrace{v'}_{\frac{v+v_*}{2}+\frac{|v-v_*|}{2}\sigma})\,d\sigma\,dv_*\,dv$$

Selecting $\phi(v)=e^{-iv\cdot\xi}$ in the identity above.

$$\mathcal{F}[Q^+(g,f)](\xi)=\int_{\mathbb{R}^3\times\mathbb{R}^3}\int_{S^2} b\left(\frac{v-v_*}{|v-v_*|}\cdot\sigma\right)g(v_*)f(v)\,e^{-i\left(\frac{v+v_*}{2}+\frac{|v-v_*|}{2}\sigma\right)\cdot\xi}\,d\sigma\,dv_*\,dv$$

$$=\int_{\mathbb{R}^3\times\mathbb{R}^3}\int_{S^2} b\left(\frac{v-v_*}{|v-v_*|}\cdot\sigma\right)g(v_*)f(v)\,e^{-i\frac{v+v_*}{2}\cdot\xi}\,e^{-i\frac{|v-v_*|}{2}\sigma\cdot\xi}\,d\sigma\,dv_*\,dv.$$

according to the general change of variable,

$$\int_{S^2} G(k\cdot\sigma,\, l\cdot\sigma)\,d\sigma=\int_{S^2} G(l\cdot\sigma,\, k\cdot\sigma)\,d\sigma,\quad |l|=|k|=1.$$

such that, we can exchange the role of $\frac{\xi}{|\xi|}$ and $\frac{v-v_*}{|v-v_*|}$.

$$\int_{S^2} g(v_*)f(v)\,b\left(\frac{v-v_*}{|v-v_*|}\cdot\sigma\right)e^{-i\frac{|v-v_*|}{2}\sigma\cdot\xi}\,d\sigma$$

$$=\int_{S^2} g(v_*)f(v)\,b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)e^{-i\frac{|\xi|}{2}\sigma\cdot(v-v_*)}\,d\sigma$$

Thus, $\mathcal{F}[Q^+(g,f)](\xi)$

$$=\int_{\mathbb{R}^3\times\mathbb{R}^3}\int_{S^2} g(v_*)f(v)\,b\left(\frac{v-v_*}{|v-v_*|}\cdot\sigma\right)e^{-i\frac{v+v_*}{2}\cdot\xi}\,e^{-i\frac{|v-v_*|}{2}\sigma\cdot\xi}\,d\sigma\,dv_*\,dv$$

$$=\int_{\mathbb{R}^3\times\mathbb{R}^3}\int_{S^2} g(v_*)f(v)\,b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)e^{-i\frac{\boxed{v}+\boxed{v_*}}{2}\cdot\xi}\,e^{-i\frac{|\xi|}{2}\sigma\cdot(\boxed{v}-\boxed{v_*})}\,d\sigma\,dv_*\,dv$$

$$=\int_{\mathbb{R}^3\times\mathbb{R}^3}\int_{S^2} g(v_*)f(v)\,b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)e^{-iv\cdot\left(\frac{\xi}{2}+\frac{|\xi|}{2}\sigma\right)}\,e^{-iv_*\cdot\boxed{\left(\frac{\xi}{2}-\frac{|\xi|}{2}\sigma\right)}}\,d\sigma\,dv_*\,dv$$

$$=\int_{S^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\hat{g}(\underbrace{\tfrac{\xi}{2}-\tfrac{|\xi|}{2}\sigma}_{\Downarrow\ \xi^-})\,\hat{f}(\underbrace{\tfrac{\xi}{2}+\tfrac{|\xi|}{2}\sigma}_{\Downarrow\ \xi^+})\,d\sigma.$$

$$\Rightarrow \mathcal{F}[Q^+(g,f)](\xi)=\int_{S^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\hat{g}(\xi^-)\hat{f}(\xi^+)\,d\sigma.$$

Exercise: $\mathcal{F}[Q^-(g,f)](\xi):=\int_{S^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\hat{g}(0)\hat{f}(\xi)\,d\sigma.$

By applying the Fourier transform to both-hand-sides of Homogeneous Boltzmann, we just get "Bobylev Identity": (*)

$$\boxed{\frac{\partial \phi(t,\xi)}{\partial t} = \int_{S^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)[\phi(t,\xi^+)\phi(t,\xi^-) - \underbrace{\phi(t,0)}_{1}\phi(t,\xi)]d\sigma.}$$

$\phi: \mathbb{R}^3 \mapsto \mathbb{C}$ is called characteristic function $\phi(\xi) = \int_{\mathbb{R}^3} e^{-iv\cdot\xi} dF(v)$

Basic property

- $\phi(0) = 1$ and $|\phi(\xi)| \leq 1$, for $\xi \in \mathbb{R}^3$.
- $\overline{\phi(\xi)} = \phi(-\xi)$, $\bar{\phi}$ denotes complex conjugate
- $\phi(\xi)$ is uniformly continuous, i.e., for all $\xi \in \mathbb{R}^3$ there exists a $\psi(\eta) \to 0$ as $|\eta| \to 0$.
  $|\phi(\xi+\eta) - \phi(\xi)| \leq \psi(\eta)$.
  Or, $|\phi(\xi+\eta) - \phi(\xi)| \leq E(|e^{-i\eta\cdot v} - 1|)$

Charactization (Bochner's Theorem) A function ϕ is called a characteristic function if and only if the following conditions hold:

(i) ϕ is a continuous function.
(ii) $\phi(0) = 1$.
(iii) ϕ is positive definite.

$\xrightarrow[\text{estimate}]{\text{give}}$
$$\begin{cases} |\phi(\xi) - \phi(\eta)|^2 \leq 2(1 - Re[\phi(\xi-\eta)]), \\ |\phi(\xi)\phi(\eta) - \phi(\xi+\eta)| \leq (1 - |\phi(\xi)|^2)(1 - |\phi(\eta)|^2) \end{cases}$$

$\xi^+ + \xi^- = \left(\frac{\xi}{2} + \frac{|\xi|}{2}\sigma\right) + \left(\frac{\xi}{2} - \frac{|\xi|}{2}\sigma\right) = \xi$

$K := \{\phi \mid \phi \text{ is characteristic function}\}$

$\boxed{K^2} := \{\phi \in K ; \|\phi - 1\|_2 < \infty\}$ with $\|\phi - 1\|_2 = \sup_{\xi \in \mathbb{R}^3} \frac{|\phi(\xi) - 1|}{|\xi|^2}$

$\updownarrow$ Fourier

$\boxed{P_2} = \{F \in P ; \int |v|^2 dF(v) < \infty\}$.

for $\phi, \psi \in K^2$

$$\|\phi - \psi\|_2 = \sup_{\xi \in \mathbb{R}^3} \frac{|\phi(\xi) - \psi(\xi)|}{|\xi|^2}$$

$$\boxed{\mathcal{F}[P_2] \subset K^2}$$

Remark: If we change $\boxed{K^\alpha}$ equipped with another norm.

$\boxed{M^\alpha}$ equipped with

$$=\{\varphi\in K;\ \|\varphi-1\|_{M^\alpha}<\infty\}$$

$$\|\varphi-1\|_{M^\alpha}=\int_{\mathbb{R}^3}\frac{|\varphi(\xi)-1|}{|\xi|^{d+\alpha}}d\xi$$

$$\|\varphi-\psi\|_{M^\alpha}=\int_{\mathbb{R}^3}\frac{|\varphi(\xi)-\psi(\xi)|}{|\xi|^{d+\alpha}}d\xi$$

$$\boxed{M^\alpha(\mathbb{R}^3)=\mathcal{F}(P_\alpha(\mathbb{R}^3))}$$

$$P_\alpha(\mathbb{R}^3)=\mathcal{F}^{-1}(M^\alpha(\mathbb{R}^3))$$

space M^α, endowed with distance $dis_{\alpha,\beta}$

$$dis_{\alpha,\beta}:=\|\varphi-\psi\|_{M^\alpha}+\|\varphi-\psi\|_\beta$$

$\beta\in[0,\alpha]$ Morimoto-Wang-Yang 2015.

Lemma: Let $\alpha\in[0,2]$. For each $\xi\in\mathbb{R}^3$, the variables ξ^+ and ξ^- are defined $\begin{cases}\xi^+=\frac{\xi}{2}+\frac{|\xi|}{2}\sigma\\ \xi^-=\frac{\xi}{2}-\frac{|\xi|}{2}\sigma\end{cases}$. then, for $\varphi\in K^\alpha$,

$$|\varphi(\xi^+)\varphi(\xi^-)-\varphi(\xi)\underbrace{\varphi(0)}_{=1}|\le 4|\xi^+|^{\frac{\alpha}{2}}|\xi^-|^{\frac{\alpha}{2}}\|\varphi-1\|_\alpha.$$

$$\boxed{\xi^++\xi^-=\xi}$$

Proposition: Assume the collisional kernel b satisfies the non-cutoff condition, for $\alpha_0\in(0,2]$. If $\varphi\in K^\alpha$ for $\alpha\in[\alpha_0,2]$, then.

$$\left|\int_{S^2}b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)[\varphi(\xi^+)\varphi(\xi^-)-\varphi(0)\varphi(\xi)]d\sigma\right|$$

$$\le\underbrace{\left(\int_0^{\frac{\pi}{2}}\sin^\alpha\left(\frac{\theta}{2}\right)b(\cos\theta)\sin\theta\,d\theta\right)}_{<\infty}\|1-\varphi\|_\alpha|\xi|^\alpha<\infty.$$

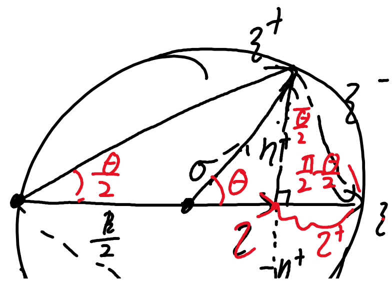
Sketch of Pf:

$$\int_{S^2}b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)[\varphi(\xi^+)\varphi(\xi^-)-\varphi(0)\varphi(\xi)]d\sigma$$

$$=\int_{S^2}b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)[\varphi(\xi^+)\varphi(\xi^-)-\varphi(\xi^+)+\varphi(\xi^+)-\varphi(\xi)]d\sigma$$

$$=\int_{S^2}b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)[\varphi(\xi^+)-\varphi(\xi)]d\sigma$$

$$+\int_{S^2}b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\varphi(\xi^+)[\varphi(\xi^-)-1]d\sigma$$



$$= \frac{1}{2}\int_{S^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\left[\phi(\xi^+)+\phi(\tilde{\xi}^+)-2\phi(\xi)\right]d\sigma \leftarrow I_1$$

$$+\int_{S^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\left[\phi(\xi)-\phi(\xi)\right]d\sigma \leftarrow I_2$$

$$+\int_{S^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\phi(\xi^+)\left[\phi(\xi^-)-1\right]d\sigma \leftarrow I_3.$$

$\boxed{\tilde{\xi}^+}$

$\eta^+ = \xi^+\sin\frac{\theta}{2}$

$|\eta^+|^\alpha = \sin^\alpha\frac{\theta}{2}\,|\xi^+|^\alpha \le \sin^\alpha\frac{\theta}{2}\,|\xi|^\alpha$

As for $\boxed{I_1}$, $\xi^+ = \xi+\eta^+$ and $\tilde{\xi}^+ = \xi+(-\eta^+)$

$$|\phi(\xi^+)+\phi(\tilde{\xi}^+)-2\phi(\xi)| = \left|\int_{\mathbb{R}^3} e^{-i\xi\cdot v}\left(e^{-i\eta^+\cdot v}+e^{i\eta^+\cdot v}-2\right)dF(v)\right|$$

$$\le \int_{\mathbb{R}^3} \underbrace{|e^{-i\xi\cdot v}|}_{\le 1}\left(2-e^{i\eta^+\cdot v}-e^{i\eta^+\cdot v}\right)dF(v)$$

$$= 2-\phi(\eta^+)-\phi(-\eta^+)$$

$$= (1-\phi(\eta^+))+(1-\phi(-\eta^+))$$

$$\le 2\|1-\phi\|_\alpha |\eta^+|^\alpha \le 2\|1-\phi\|_\alpha |\xi|^\alpha \sin^\alpha\left(\frac{\theta}{2}\right).$$

\#.

Important parameters:

(I) Under cutoff assumption b_c : $\int_{S^2} b_c\, d\sigma < \infty$

for all $\alpha\in[0,2]$ and $\xi\in\mathbb{R}^3\setminus\{0\}$

$$\nu_\alpha := \int_{S^2} b_c\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\frac{|\xi^+|^\alpha+|\xi^-|^\alpha}{|\xi|^\alpha}\,d\sigma$$

$$= 2\pi\int_0^\pi b_c(\cos\theta)\left(\sin^\alpha\frac{\theta}{2}+\cos^\alpha\frac{\theta}{2}\right)\sin\theta\, d\theta$$

$$\overset{\cos\theta := s}{=} 2\pi\int_{-1}^{1} b_c(s)\left[\left(\frac{1+s}{2}\right)^{\frac{\alpha}{2}}+\left(\frac{1-s}{2}\right)^{\frac{\alpha}{2}}\right]ds$$

is finite and independent of ξ.

$$\nu_\alpha > \nu_2 = \int_{S^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)d\sigma = 2\pi\int_{-1}^{1} b_c(s)\, ds.$$

(II) Under Non-cutoff assumption. $\int_{S^2} b\, d\sigma = \infty$.

$$\lambda_\alpha = \int_{S^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\left(\boxed{\frac{|\xi^+|^\alpha+|\xi^-|^\alpha}{|\xi|^\alpha}}-1\right)d\sigma$$

is finite, independent of ξ, and positive provided $0<\alpha<2$.

key point: for $-1<s<1$

$$c(1-s^2)^{\frac{\alpha}{2}} \le \left[\left(\frac{1+s}{2}\right)^{\frac{\alpha}{2}} + \left(\frac{1-s}{2}\right)^{\frac{\alpha}{2}} - 1\right] \le C\underbrace{(1-s^2)}_{(1-s)(1+s)}{}^{\frac{\alpha}{2}}$$

Well-posedness under cutoff:

$$\partial_t \varphi(t,\xi) = \hat{Q}(\varphi,\varphi) = \Gamma[\varphi] = \Gamma^+[\varphi] - \Gamma^-[\varphi] \Rightarrow \nu_2\varphi(\xi)$$

$$\int_{S^2} b_c\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\varphi(\xi)\,d\sigma$$

$$\Rightarrow \partial_t \varphi(t,\xi) + \nu_2 \varphi(t,\xi) = \Gamma^+[\varphi] \qquad \boxed{\frac{\partial u}{\partial t} + Au = \Gamma[u]}$$

$$\Gamma^+[\varphi] = \int_{S^2} b_c\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\varphi(\xi^+)\varphi(\xi^-)\,d\sigma$$

$$\Rightarrow \boxed{\varphi(t,\xi) = \varphi_0 e^{-\nu_2 t} + \int_0^t e^{-\nu_2(t-\tau)}\Gamma^+[\varphi](\tau,\xi)\,d\tau} \quad \checkmark$$

$$\Downarrow$$

$$P[\varphi]$$

Lemma: let $\alpha \in [0,2]$, b_c is under cutoff assumption.

① $\Gamma^+[\varphi]$ is continuous and positive definite. $\quad$ Γ^+ maps $K \mapsto K$.

Moreover, for $\varphi, \tilde\varphi \in K^\alpha$,

② $|\Gamma^+[\varphi](\xi) - \Gamma^+[\tilde\varphi](\xi)| \le \nu_\alpha \|\varphi - \tilde\varphi\|_\alpha |\xi|^\alpha$

Pf: $\boxed{\text{For ②.}}$ $|\Gamma^+[\varphi] - \Gamma^+[\tilde\varphi]|$ $\qquad$ $\boxed{|\varphi^+|, |\varphi^-| < 1}$

$$= \left|\int_{S^2} b_c\left[(\varphi^+ - \tilde\varphi^+)\varphi^- + \tilde\varphi^+(\varphi^- - \tilde\varphi^-)\right]d\sigma\right|$$

$$\le \int_{S^2} b_c\left[\|\varphi-\tilde\varphi\|_\alpha |\xi^+|^\alpha + \|\varphi-\tilde\varphi\|_\alpha |\xi^-|^\alpha\right]d\sigma$$

$$= \|\varphi-\tilde\varphi\|_\alpha \nu_\alpha |\xi|^\alpha$$

$\boxed{\text{For ①}}$: Since $\Gamma^+[\varphi,\tilde\varphi]$ is continuous at $\xi=0$, it is sufficient to prove that it can be defined as the pointwise limit of characteristic functions.

$$\int_{S^2} b_c\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\varphi(\xi^+)\tilde\varphi(\xi^-)\,d\sigma$$

$$\Gamma_m^+(\xi) = \Gamma^+[\varphi,\tilde\varphi](\xi)\,e^{-\frac{1}{2m}|\xi|^2}$$

$$= \int_{S^2} b_c\left(\frac{\xi\cdot\sigma}{|\xi|}\right) \underline{\phi(\xi^+) e^{-\frac{1}{2m}|\xi^+|^2}}\; \underline{\tilde{\phi}(\xi^-) e^{-\frac{1}{2m}|\xi^-|^2}}\, d\sigma$$

Now the sequence $\{P_m^+\}$ converge pointwise to $\{P^+\}$

let F and $\tilde{F}$ be the probability measures, corresponding to ϕ and $\tilde{\phi}$ respectively,

$$\begin{cases} \phi(\xi) := \int_{\mathbb{R}^3} e^{-i v\cdot\xi} dF(v) \\ \tilde{\phi}(\xi) := \int_{\mathbb{R}^3} e^{-i v\cdot\xi} d\tilde{F}(v) \end{cases}$$

Then $\boxed{\phi(\xi^+) e^{-\frac{1}{2m}|\xi^+|^2}}$ and $\boxed{\tilde{\phi}(\xi^-) e^{-\frac{1}{2m}|\xi^-|^2}}$ are the characteristic function with respect to

$$\begin{cases} f_m(v) = \int_{\mathbb{R}^3} w_m(v-u)\, dF(u) \\ \tilde{f}_m(v) = \int_{\mathbb{R}^3} w_m(v-u)\, d\tilde{F}(u) \end{cases}$$

where $w_m(v) = \frac{1}{(2\pi \frac{1}{m})^{\frac{3}{2}}} e^{-\frac{m v^2}{2}}$

Since $\phi(\xi^+) e^{-\frac{1}{2m}|\xi^+|^2}$ and $\tilde{\phi}(\xi^-) e^{-\frac{1}{2m}|\xi^-|^2}$ belong to $L^2(\mathbb{R}^3)$

f_m and $\tilde{f}_m$ can be determed. On the other hand,

$$P_m^+[\phi,\tilde{\phi}]^\vee(t,v) = \int_{\mathbb{R}^3}\int_{S^2} b_c\left(\frac{\xi\cdot\sigma}{|\xi|}\right) f_m(t,v')\, g_m(t,u')\, d\sigma\, du.$$

is a probability density function as well.

$$P_m^+[\phi,\tilde{\phi}] = \int_{\mathbb{R}^3} P^+[f_m, \tilde{f}_m] e^{-i v\cdot\xi} dv.$$

Then, P_m^+ is a characteristic function, which converges pointwise to $P^+[\phi,\tilde{\phi}]$. #

Theorem: (Wellposed under cutoff assumption)

Assume $\alpha \in [0,2]$ and b_c satisfy cutoff assumption.

Then. for $\phi_0(\xi) \in K^\alpha$, there exists a unique solution $\phi(t,\xi)$ such that $\phi \in \mathcal{X}^\alpha := C([0,\infty), K^\alpha)$

Sketch of Proof: $\mathcal{X}_T^\alpha := \{\phi \in C([0,T], K^\alpha);\ \sup_{t\in[0,T]} \|\phi(t,\cdot)\|_\alpha < \infty\}$

is a complete metric space w.r.t

$\|\cdot\|_{\mathcal{X}^\alpha} := \sup \|\cdot\|$

$\|\cdot\|_{X_T^\alpha} = \sup_{t\in[0,T]} \|\cdot\|_\alpha$.

① $P[\phi] \in X_T^\alpha$: $\|P[\phi]-1\|_{X_T^\alpha} < \infty$

② $|P[\phi]-P[\tilde{\phi}]| \leq \gamma_2 T \|\phi-\tilde{\phi}\|_{X_T^\alpha} |\xi|^\alpha$

$\boxed{T \leq \frac{1}{\gamma_2}}$

Theorem: (Wellposedness under Non-Cutoff Assumption)

Assume b satisfy the Non-Cutoff condition with $\alpha_0 \in [0,2]$. For $\alpha \in [\alpha_0, 2]$ and initial datume $\phi_0 \in K^\alpha$, there exists a solution $\phi \in C([0,\infty), K^\alpha)$ and solution ϕ is unique in the space $C([0,\infty), K^{\alpha_0})$.

Sketch of Proof: $b_n(s) = \min\{b(s), n\} \leq b(s)$, $n \in \mathbb{N}$.

By apply Cutoff Theorem. we can find sequence $\{\phi_n\} \in C([0,\infty), K^\alpha)$. $\xrightarrow{\text{Ascoli-Arzela}} \{\phi\} \in C([0,\infty), K^\alpha)$

① Uniform Bounded: $|\phi_n| < 1$

② Equi-continuity in "t": $|\partial_t \phi_n(t,\xi)| \lesssim e^{\lambda_\alpha t} \|\phi_0 - 1\| |\xi|^\alpha$

③ Equi-continuity in "ξ": $|\phi_n(t,\xi) - \phi_n(t,\eta)| \lesssim |\xi-\eta|^{\frac{\alpha}{2}} \|\phi_0 - 1\|_\alpha^{\frac{1}{2}}$

#

$\phi_0 := \int e^{-iv\cdot\xi} dF_0(v) \in K^\alpha$

Theorem (Smoothing effect of measure-valued solution)

Let b satisfy Non-Cutoff assumption with $0<\alpha_0<2$ and $\alpha \in (\alpha_0, 2]$. If $F_0(v) \in \tilde{P}_\alpha(\mathbb{R}^3) \supset P_\alpha$ is NOT a single Dirac measure, So $F^{-1}(K^\alpha)$ (by the wellposed Result) $F(t,v)$ is a unique solution in $C([0,\infty), \tilde{P}_\alpha)$.

Then, there exists a $T>0$ such that $F(t,\cdot) \in H^\infty(\mathbb{R}^3)$ for $0<t\leq T$. Moreover, if $F_0(v) \in P_2(\mathbb{R}^3)$, then $T=\infty$.

$F \Leftrightarrow \phi$

Lemma (Degenerate Coercivity Estimate) ✓

Let $F_0 \in \tilde{P}_2(\mathbb{R}^3)$ and $F_t(v) \in C([0,\infty), \tilde{P}_2)$ is obtained in the Well-posedness Theorem. If $\phi(t,\xi)$ and $\phi_0(\xi)$ is the Fourier transform of $F_t(v)$ and $F_0(v)$ respectively, then there exists $T>0$ and $C>0$, such that for $t \in [0,T]$, we have

$$t\int_{\mathbb{R}^3} \langle\xi\rangle^{2s} |h(\xi)|^2 d\xi \le C\Big(\int_{\mathbb{R}^3}\Big(\int_{S^2} b\big(\tfrac{\xi\cdot\sigma}{|\xi|}\big)\big(1-|\phi(t,\xi^-)|\big)d\sigma\Big)|h(\xi)|^2 d\xi \quad (**)$$

$\|h\|^2_{L^2_s}$

$$+\int_{\mathbb{R}^3}|h(\xi)|^2 d\xi.\Big)$$

for all $h(\xi) \in L^2_s$ and $\xi^- = \frac{\xi - |\xi|\sigma}{2}$.

Proof of Smoothing Effect Theorem: We first admit Lemma above.

$$\begin{cases} \partial_t\phi(t,\xi) = \int_{S^2} b\big(\frac{\xi\cdot\sigma}{|\xi|}\big)[\phi(t,\xi^+)\phi(t,\xi^-) - \phi(t,\xi)\phi(t,0)]d\sigma \\ \phi(t=0,\xi) = \phi_0(\xi), \text{ with } \xi^\pm = \frac{\xi}{2} \pm \frac{|\xi|}{2}\sigma \end{cases} \quad (*)$$

$$\|u\|_{H^s(\mathbb{R}^3)} := \|(1+|\xi|^s)\hat{u}(\xi)\|_{L^2} = \Big(\int_{\mathbb{R}^3} |\underbrace{(1+|\xi|^s)}_{\langle\xi\rangle^s}|^2 |\hat{u}(\xi)|^2 d\xi\Big)^{\frac{1}{2}} \le \infty$$

$\hat{u} \to F$, $\hat{u} \to \phi$

By Well-posedness Theorem, $\phi \in C([0,\infty), K^\alpha)$

Define a time-dependent weight function: $\sup_{\xi\in\mathbb{R}^3} \frac{|\phi-1|}{|\xi|^\alpha} < \infty$

$$\boxed{M_\delta(t,\xi) = \langle\xi\rangle^{Nt^2-4}\langle\delta\xi\rangle^{-2N_0}}, \quad \langle\xi\rangle^2 = 1+|\xi|^2.$$

where $N_0 = NT^2/2 + 2$, $N \in \mathbb{N}$ and $\delta > 0$.

Let's multiply the Eq. $(*)$ by $M_\delta^2(t,\xi)\overline{\phi(t,\xi)}$ and integrate with respect to ξ over $\mathbb{R}^3$. The R-H-S becomes.

$$-2\,\mathrm{Re}[(\phi^+\phi^- - \phi)M_\delta^2\bar{\phi}] \qquad -ab \ge -\Big(\frac{a^2}{2}+\frac{b^2}{2}\Big)$$

$$= \underbrace{(1|M_\delta\phi|^2 + |M_\delta^+\phi^+|^2 - 2\,\mathrm{Re}[\phi^-(M_\delta^+\phi^+)\overline{M_\delta\phi}]}_{(J_1)}$$

$$+\underbrace{\left(|M_\delta\phi|^2-|M_\delta^+\phi^+|^2\right)}_{(J_2)}+\underbrace{2Re\left[\phi^-(M_\delta-M_\delta^+)\phi^+)\overline{M_\delta\phi}\right]}_{(J_3)}$$

ξ^+ $\qquad \xi\mapsto\xi^+$ $\qquad \sin^2\frac{\theta}{2}M^+\phi^+$

$\Rightarrow v\mapsto v^+=\frac{v}{2}+\frac{|v|}{2}\sigma$

As for (J_1) term, by applying the Cauchy inequality to $\left[-2Re[\phi^-(M_\delta^+\phi^+)\overline{M_\delta\phi}]\right]$, and find that

$$J_1\geq\boxed{(1-|\phi^-|)}(|M_\delta\phi|^2+|M_\delta^+\phi^+|^2)\geq(1-|\phi^-|)|M_\delta\phi|^2.$$

Therefore, considering (**) in the Lemma.

$$\int_{\mathbb{R}^3}\int_{S^2}b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)J_1\,d\sigma d\xi+\int_{\mathbb{R}^3}|M_\delta\phi|^2d\xi\gtrsim t\int_{\mathbb{R}^3}\langle\xi\rangle^{2s}|M_\delta\phi|^2d\xi$$

$A\geq C_0B.$

As for (J_2) term, we have to apply the change of variable $\xi\mapsto\xi^+$ in $M_\delta^+\phi^+$.

Lemma 1, A-D-V-W2000

Cancellation Lemma.

$$\left|\int_{\mathbb{R}^3}\int_{S^2}b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)J_2\,d\sigma d\xi\right|=2\pi\left|\int_{\mathbb{R}^3}|M_\delta\phi|^2\left(\int_0^{\frac{\pi}{2}}b(\cos\theta)\sin\theta\left(1-\frac{1}{\cos^3\frac{\theta}{2}}\right)d\theta\right)d\xi\right|$$

θ

$$\lesssim\int_{\mathbb{R}^3}|M_\delta\phi|^2d\xi.$$

As for (J_3) term,

$$|M_\delta-M_\delta^+|\lesssim\sin^2\frac{\theta}{2}M_\delta^+$$

$$\left|\int_{\mathbb{R}^3}\int_{S^2}b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\cdot J_3\,d\sigma d\xi\right|\lesssim\int_{\mathbb{R}^3}|M_\delta\phi|^2d\xi.$$

Note L-H-S is:

$$\frac{|\xi|^{2s}}{\log\langle\xi\rangle}\to\infty,\ \text{as } |\xi|\to\infty$$

$$2Re\left(\frac{\partial\phi}{\partial t}M_\delta^2\bar{\phi}\right)=\frac{\partial|M_\delta\phi|^2}{\partial t}-4N+\log\langle\xi\rangle|M_\delta\phi|^2.$$

Combining the estimates of (J_1), (J_2) and (J_3),

$$\frac{d}{dt}\int_{\mathbb{R}^3}|M_\delta(t,\xi)\phi(t,\xi)|^2d\xi\lesssim\int_{\mathbb{R}^3}|M_\delta(t,\xi)\phi(t,\xi)|^2d\xi$$

which implies for $t\in[0,T]$, "F" $\in H^\infty$

$$\int_{\mathbb{R}^3}\left|\langle\xi\rangle^{Nt^2-4}(1+\delta|\xi|^2)^{-N_0}\phi(t,\xi)\right|^2d\xi\lesssim\int_{\mathbb{R}^3}|\langle\xi\rangle^{-4}\phi_0(\xi)|^2d\xi$$

letting $\delta \to 0$, we can prove $F \in H^{\infty}(0,T)$, because we can take arbitrarily Large N.

For the case of $F_0(v) \in P_2(\mathbb{R}^3)$. We notice that the energy of solution is uniformly bounded by $\|F_0\|_{L^1_2}$, i.e.,

$$\int_{\mathbb{R}^3} |v|^2 dF_T(v) \le \int_{\mathbb{R}^3} |v|^2 dF_0(v) \text{ for } T>0.\quad f(T,v) \in L^{\infty}(\mathbb{R}^3)$$

$$\Rightarrow \|f(T)\|_{L\log L} := \int_{\mathbb{R}^3} \log(1+f(T,v)) f(T,v) dv < \infty.$$

$$\Rightarrow f(T) \in L^1_2 \cap L\log L. \quad \text{Villani 98}$$

$$\Rightarrow \sup_{t \ge T} \left(\|f(t)\|_{L^1_2} + \|f\|_{L\log L} \right) < \infty$$

which implies that the exist a $k>0$ independent of t such that

$$\boxed{\underbrace{1 - |\varphi(t,\xi)|}_{\int_{\mathbb{R}^3} 1\, dF_t(v) - \int e^{-iv\cdot\xi} dF_t(v)} \ge k \min\{1, |\xi|^2\}.} \quad \text{lemma3 A-D-V-W 2000.}$$

Therefore, for $|\xi| \ge R$ for some $R>0$ large enough,

$$\int_{S^2} b\left(\tfrac{\xi}{|\xi|}\cdot\sigma\right)\left(1-|\varphi(t,\xi^-)|\right) d\sigma \ge 2\pi k \int_0^{|\xi|^{-1}} b(\cos\theta) |\xi|^2 \sin\theta \, d\theta$$

$$\gtrsim |\xi|^2 \int_0^{|\xi|^{-1}} \theta^{1-2s} d\theta \qquad \theta^{2-2s}\Big|_0^{|\xi|^{-1}}$$

$$\gtrsim |\xi|^{2s} \qquad\qquad = |\xi|^{2s-2}$$

$$\int_{\mathbb{R}^3} \left(\int_{S^2} b\left(\tfrac{\xi}{|\xi|}\cdot\sigma\right)\left(1-|\varphi(t,\xi^-)|\right) d\sigma \right) d\xi \gtrsim \cancel{\times} \int_{\mathbb{R}^3} |\xi|^{2s} |\varphi|^2 d\xi.$$

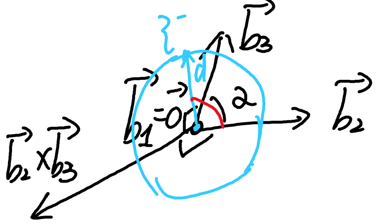
which gives the standard coercivity estimate, and it leads to $F(t,v) \in H^{\infty}(\mathbb{R}^3)$ for $\forall t > T$.

Proof the lemma (Degenerate Coercivity Estimate) (Microlocal analysis)

In the case of Initial Condition $F_0(v)$ not Concentrate on a straight line i.e., we can assume that there exist three small balls $A_i = B(b_i, \delta)$ with Center $b_i = v$ and radius $\delta > 0$ such that

$$\int_{A_i} dF_0(v) = m_i > 0, \quad i = 1,2,3$$

By linear transformation, $\vec{b_1}=\vec{0}$, $\vec{b_2}$ and $\vec{b_3}$ are linearly independent.



$$\eta_0 = 1 - \left|\frac{\vec{b_2}}{|\vec{b_2}|}\cdot\frac{\vec{b_3}}{|\vec{b_3}|}\right| = 1-|\cos\alpha|$$

Take two positive constants $d_1<d_2$ such that

$$0<d_1\min\{|\vec{b_2}|,|\vec{b_3}|\}<d_2\max\{|\vec{b_2}|,|\vec{b_3}|\}\le\frac{\pi}{2}.$$

let $d=\frac{d_1+d_2}{2}$. Then, we assume $\boxed{\xi^-}$ varies on the circle,

$$C=\{\xi\in\mathbb{R}^3;\ |\xi|=d,\ \xi\perp(\vec{b_2}\times\vec{b_3})\}$$

Denote $\boxed{\int_{A_j} e^{-iv\cdot\xi^-}\,dF(v) = m_j(a_j+ib_j)}$, $j=1,2,3$.

Since $|a_j+ib_j|\le 1$,

$$\begin{cases}(a_1,b_1)=(1,0)+\vec{e_1}\\ (a_2,b_2)=(\cos(|\xi^-||\vec{b_2}|\cos\nu_1),\ \sin(|\xi^-||\vec{b_2}|\cos\nu_1))+\vec{e_2}\\ (a_3,b_3)=(\cos(|\xi^-||\vec{b_3}|\cos\nu_2),\ \sin(|\xi^-||\vec{b_3}|\cos\nu_2))+\vec{e_3}\end{cases}$$

ν_1 angle between $\vec{b_2}$ and ξ^-

ν_2 angle between $\vec{b_3}$ and ξ^-.

$\nu_2=\nu_1\pm\alpha$

For δ is small enough,

$$2-\left|\frac{(a_1,b_1)}{|(a_1,b_1)|}\cdot\frac{(a_2,b_2)}{|(a_2,b_2)|}\right|-\left|\frac{(a_1,b_1)}{|(a_1,b_1)|}\cdot\frac{(a_3,b_3)}{|(a_3,b_3)|}\right|$$

$$=2-\cos(|\xi^-||\vec{b_2}|\cos\nu_1)-\cos(|\xi^-||\vec{b_3}|\cos(\nu_1\pm\alpha))-O(\delta)\ge C_0\eta_0$$

If $\phi_0(\xi)=\int_{\mathbb{R}^3}e^{-iv\cdot\xi}dF_0(v)$ and ξ^- varied C,

$$\phi_0(0)-|\phi_0(\xi^-)|=1-\left|\int_{A^c\cup\bigcup_{j=1}^{3}A_j}e^{-iv\cdot\xi^-}dF_0(v)\right|$$

$$\ge\sum_{j=1}^{3}\underbrace{\int_{A_j}dF_0(v)}_{m_j}-\left|\sum_{j=1}^{3}\int_{A_j}e^{-iv\cdot\xi^-}dF_0(v)\right|$$

$$=\sum_{j=1}^{3}m_j-\left|\sum_{j=1}^{3}m_j(a_j+ib_j)\right|$$

$$\ge\min\{m_1,m_2,m_3\}\left(3-\left|\sum_{j=1}^{3}(a_j+ib_j)\right|\right)$$

$$-3\min\{m_1,m_2,m_3\}\left\{2-\left|\frac{(a_1,b_1)}{|(a_1,b_1)|}\cdot\frac{(a_2,b_2)}{|(a_2,b_2)|}\right|-\left|\frac{(a_2,b_2)}{|(a_2,b_2)|}\cdot\frac{(a_3,b_3)}{|(a_3,b_3)|}\right|\right\}$$
$$\geq \frac{1}{3}\underbrace{\min\{m_1,m_2,m_3\}\, C_0\eta_0}\; := \underline{K_0}.$$

Since $|a_j+ib_j|\leq 1$ and

$$\left|\sum_{j=1}^{3}(a_j+ib_j)\right|^2 \leq \left(\underbrace{|a_1+ib_1|}_{\leq 1}+\sum_{j=2}^{3}\underbrace{|a_j+ib_j|}_{\leq 1}\left|\frac{(a_1,b_1)}{|a_1,b_1|}\cdot\frac{(a_j,b_j)}{|a_j,b_j|}\right|\right)^2$$
$$+\left(\sum_{j=2}^{3}\underbrace{|a_j+ib_j|}_{\leq 1}\left|\frac{(a_1,b_1)}{|a_1,b_1|}\times\frac{(a_j,b_j)}{|a_j,b_j|}\right|\right)^2$$
$$\leq\left(1+\sum_{j=2}^{3}\left|\frac{(a_1,b_1)}{|(a_1,b_1)|}\cdot\frac{(a_j,b_j)}{|(a_j,b_j)|}\right|\right)^2+\left(\sum_{j=2}^{3}\left|\frac{(a_1,b_1)}{|a_1,b_1|}\times\frac{(a_j,b_j)}{|a_j,b_j|}\right|\right)^2$$
$$\leq 5+2\sum_{j=2}^{3}\left|\frac{(a_1,b_1)}{|(a_1,b_1)|}\cdot\frac{(a_j,b_j)}{|(a_j,b_j)|}\right|.$$

For $\Phi(t,\xi)$ and $\Phi_0(\xi)$, there exist $\mu>0$, $\varepsilon>0$, $T>0$.

$$\boxed{\xi^-}\in\boxed{C_{\mu,\varepsilon}}=\left\{\eta\in\mathbb{R}^3;\ d-\mu\leq|\eta|\leq d+\mu,\ \underbrace{\left|\frac{\eta}{|\eta|}\cdot\left(\frac{\vec{b_2}\times\vec{b_3}}{|\vec{b_2}\times\vec{b_3}|}\right)\right|\leq\varepsilon}\right\}$$

we have $1-|\Phi(t,\xi^-)|\geq\frac{K_0}{2}$ for $t\in[0,T]$

Take a $R>0$ such that $\boxed{\frac{d+\mu}{R}=\frac{\varepsilon}{10}.}$ Let $|\xi|\geq R$ and $\underline{w=\frac{\xi}{|\xi|}}\in S^2$

take $\sigma=(\cos\theta,\sin\theta\cos\phi,\sin\theta\sin\phi)$ with $\begin{cases}\theta\in[0,\frac{\pi}{2}]\\ \phi\in[0,2\pi]\end{cases}$

$\xi^-=\frac{\xi}{2}-\frac{|\xi|}{2}\sigma=\xi^-(\theta,\phi)$

If θ satisfies $d-\mu\leq\underline{|\xi^-(\theta,\phi)|=|\xi|\sin\frac{\theta}{2}}\leq d+\mu$.

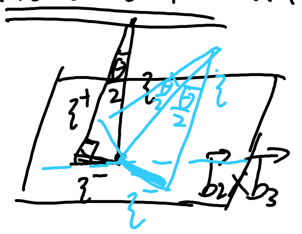
then there exists an interval $\underline{I_w\subset[0,2\pi]}$ such that $\xi^-(\theta,\phi)\in C_{\mu,\varepsilon}$

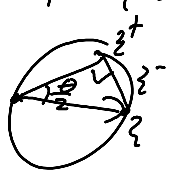
Since $\frac{\theta}{2}\leq\sin^{-1}\left(\frac{d+\mu}{R}\right)<\frac{\varepsilon}{5}$.

and the set

$\{\lambda\xi^-(\theta,\phi)\in\mathbb{R}^3;\ \phi\in[0,2\pi],\ 0<\lambda\leq 1\}$

$\underline{\text{intersected}}$ with the plane $\text{span}\{\vec{b_2},\vec{b_3}\}$.





Therefore, for any ξ belong to conic nearby w,

$$\Gamma_w = \left\{ \xi \in \mathbb{R}^3 ; \left| \frac{\xi}{|\xi|} - w \right| < \underset{\text{sufficiently small.}}{\varepsilon_w} , |\xi| \geq R \right\}$$

then

$$\int_{\mathbb{R}^3} \left(\int_{S^2} b\left(\frac{\xi}{|\xi|}\cdot\sigma\right) \left(1 - |\phi(t,\xi^-)|\right) d\sigma \right) |h(\xi)|^2 d\xi.$$

$$\gtrsim \int_{\xi \in \Gamma_w} \int_{\phi \in I_w} \left(\int_{2\sin^{-1}(d-\mu)/|\xi|}^{2\sin^{-1}(d+\mu)/|\xi|} \theta^{-1-2s} \frac{k_0}{2} d\theta \right) d\phi \, |h(\xi)|^2 d\xi$$

$$\gtrsim \int_{\Gamma_w} |\xi|^{2s} |h(\xi)|^2 d\xi.$$

which implied that, with the help of covering argument on S^2,

$$\int_{\mathbb{R}^3} \left(\int_{S^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right) \left(1 - |\phi(t,\xi^-)|\right) d\sigma \right) |h(\xi)|^2 d\xi + \int_{\mathbb{R}^3} |h(\xi)|^2 d\xi$$

$$\gtrsim \int_{\mathbb{R}^3} \langle \xi \rangle^{2s} |h(\xi)|^2 d\xi. \quad \text{if } t \in [0,T]. \quad \#$$