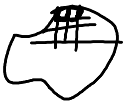
Lecture_7

Thursday, 21 April 2022   2:33 pm

Numerical Method for the Homogeneous Boltzman Equation:

① Finite Difference $\quad \partial_t f \approx \frac{f^{n+1}-f^n}{\Delta t}$

② Finite Element 

③ Spectral Method

"Fourier"

1. Main idea of Spectral Method. $\{1, \cos x, \sin x, \cos 2x, \cdots\}$

$$u \in L^2[0,2\pi] \Rightarrow F[u] = \hat{a}_0 + \sum_{n=1}^{\infty} \hat{a}_n \cos nx + \sum_{n=1}^{\infty} \hat{b}_n \sin nx.$$

where the expansion coefficients are

$$\hat{a}_n = \frac{1}{C_n \pi} \int_0^{2\pi} u(x) \cos(nx)\, dx \qquad \hat{b}_n = \frac{1}{\pi} \int_0^{2\pi} u(x) \sin(nx)\, dx$$

with the value $C_n = \begin{cases} 2, & n=0 \\ 1, & n>0 \end{cases}$

$$F[u] = \sum_{|n| \le \infty} \hat{u}_n e^{inx}$$

where

$$\hat{u}_n = \frac{1}{2\pi} \int_0^{2\pi} u(x) e^{-inx}\, dx = \begin{cases} \hat{a}_0, & n=0 \\ \frac{\hat{a}_n - i\hat{b}_n}{2}, & n>0 \\ \frac{\hat{a}_n + i\hat{b}_n}{2}, & n<0. \end{cases}$$

Remark: (1) $u(x)$ is real-valued, $\hat{u}_{-n} = \overline{\hat{u}_n}$

(2) $u(x)$ is real and even, "$u(x) = u(-x)$", then $\hat{b}_n = 0$

(3) $u(x)$ is real and odd, "$u(x) = -u(-x)$", then $\hat{a}_n = 0$.

? whether the truncated series has some "good" property?

$$\boxed{P_N u(x) = \sum_{|n| \le N/2} \hat{u}_n e^{inx}}$$

is projection to a finite-dimensional space

$\mathbb{P}_N = \mathrm{span}\{ e^{inx} \mid |n| \le \frac{N}{2} \}$ $\quad \mathrm{Dim}(\mathbb{P}_N) = N+1$

Theorem: If the sum of squares of Fourier coefficients is bounded $\sum_{|n|\le\infty} |\hat{u}_n|^2 < \infty$.

Then, the truncated series converges in the L^2 norm:

$$\boxed{\|u - P_N u\|_{L^2[0,2\pi]} \to 0 \quad \text{as } N \to \infty.}$$

If, moreover, the sum of absolute values of the Fourier coefficients is bounded $\sum_{|n|<\infty} |\hat{u}_n| < \infty$,

Then, the truncated series converges uniformly.

$$\|u - P_N u\|_{L^\infty[0,2\pi]} \to 0 \quad \text{as } N \to \infty.$$

? How about the Differentiation of continuous expansion?

$$\frac{\partial u}{\partial x} \overset{?}{\Longleftrightarrow} \frac{\partial P_N u}{\partial t}$$

$P_N u(x) = \sum_{|n|\le\frac{N}{2}} \hat{u}_n e^{inx}$, we can obtain

$$\frac{d^q}{dx^q} P_N u(x) = \sum_{|n|\le\frac{N}{2}} \hat{u}_n \frac{d^q}{dx^q} e^{inx} = \sum_{|n|\le\frac{N}{2}} (in)^q \hat{u}_n e^{inx} = P_N \frac{d^q}{dx^q} u$$

$\Longrightarrow$ Projection and Differentiation operator commute.

$\overset{u(t,x)}{\Longrightarrow}$ $\boxed{u_t = \mathcal{L} u} \Rightarrow (P_N u)_t = P_N \mathcal{L} u = \mathcal{L} P_N u.$ $\quad P_N u \overset{?}{\Longleftrightarrow} u_N$

$\|u_N - u\|_X$

Example 1:
$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \mathcal{L} u(t,x), \quad t\ge 0,\ x\in[0,2\pi]. \\ u(t=0,x) = g(x) \end{cases}$$

We seek $u_N(t,x)$ from the $\mathbb{P}_N \subset \text{span}\{e^{inx}\}_{|n|\le\frac{N}{2}}$.

$$u_N(t,x) = \sum_{|n|\le\frac{N}{2}} \boxed{a_n(t)} e^{inx}$$

where $\{a_n(t)\}$ will be determined by the requirement that the residual $R_N(t,x) = \dfrac{\partial u_N(t,x)}{\partial t} - \mathcal{L} u_N(t,x)$ "small" is orthogonal to $\mathbb{P}_N$

$$\Rightarrow \|R(t,x)\| = \sum_{|n|\le\infty} R_n e^{\cdots}$$

$$\Rightarrow \boxed{R_n' = \frac{1}{2\pi}\int_0^{2\pi} R_N(t,x)e^{-inx}dx = 0}, \quad \forall |n| \le \frac{N}{2}.$$

$\Rightarrow$ "N+1 ODE to determine (N+1) $a_n(t)$"

$$\begin{cases} u_N(t=0,x) = \sum_{|n|\le\frac{N}{2}} a_n(0)e^{inx} \\ a_n(0) = \frac{1}{2\pi}\int_{-1}^{1} g(x)e^{-inx}dx. \end{cases}$$

Example 2: $\begin{cases} \frac{\partial u(t,x)}{\partial t} = c\frac{\partial u(t,x)}{\partial x} + \varepsilon\frac{\partial^2 u(t,x)}{\partial x^2} \\ u(t=0,x) = u_0(x) \in C_p^\infty[0,2\pi] \end{cases}$

We seek $u_N(t,x) = \sum_{|n|\le\frac{N}{2}} a_n(t)e^{inx}$

$$\Rightarrow R_N(t,x) = \frac{\partial u_N(t,x)}{\partial t} - c\frac{\partial u_N(t,x)}{\partial x} - \varepsilon\frac{\partial^2 u_N(t,x)}{\partial x^2}$$

orthogonal to $\mathbb{P}_N = \text{span}\{e^{inx}\}_{|n|\le\frac{N}{2}}$

Recall that $\frac{\partial u_N(t,x)}{\partial x} = \sum_{|n|\le\frac{N}{2}} (in)a_n(t)e^{inx}$

$$\Rightarrow R_N(t,x) = \sum_{|n|\le\frac{N}{2}}\left(\frac{da_n(t)}{dt} - cin\,a_n(t) + \varepsilon n^2 a_n(t)\right)e^{inx}$$

$\xrightarrow[\text{to } \mathbb{P}_N]{\text{Project } R_N}$ $\boxed{\frac{da_n(t)}{dt} = (cin - \varepsilon n^2)a_n(t)}, \quad \forall |n| \le \frac{N}{2}.$

2. Apply (Fourier) Spectral Method to Boltzmann Equation

$$\begin{cases} \frac{\partial f(t,v)}{\partial t} = \boxed{Q(f,f)} = \int_{\mathbb{R}^3}\int_{S^2} \underbrace{B(|v-v_*|,\sigma)}_{b(\cos\theta)|v-v_*|}(f_*'f' - f_*f)d\sigma dv_* \\ f(t=0,v) = f_0(v) \end{cases}$$

$v \in \mathbb{R}^3$

Computation domain

Goal: Seek $f_N(v) = \sum_{k=-N}^{N} \boxed{\hat{f}_k(t)} e^{ik\cdot v}$.

Step I: Preparation $\boxed{q = v - v_*}$

$$Q(f,f)(v) = \int_{\mathbb{R}^3}\int_{S^2} B(|q|,\sigma)[f(v+q^+)f(v+q^-) - f(v)f(v+q)]d\sigma dq$$

where $\begin{cases} q^+ = \frac{1}{2}(q + |q|\sigma) \\ q^- = \frac{1}{2}(q - |q|\sigma) \end{cases}$

$$\int_{\mathbb{R}^3} Q(f,f)(v)\,\phi(v)\,dv$$

$$= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} B(|q|,\sigma)\, f(v) f(v+q)\left[\phi(v+q^+) - \phi(v)\right] d\sigma\, \boxed{dq}\, \boxed{dv}$$

Step II: Periodization and Choice of Integration Domain.

Proposition: let $\mathrm{supp}(f(v)) \subset B_0(R)$. Then

(i) $\mathrm{supp}(Q(f,f)(v)) \subset B_0(\sqrt{2}R)$

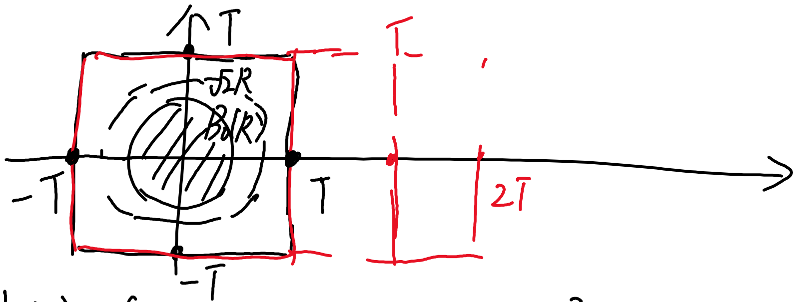
$f_N(v)\, f_N(v+q)$

(ii) $\displaystyle\int_{\mathbb{R}^3} Q^R(f,f)\,\phi(v)\,dv = \int_{B_0(\sqrt{2}R)}\int_{B_0(2R)}\int_{S^2} B(|q|,\sigma) f(v) f(v+q)$

$[\phi(v+q^+) - \phi(v)]\, d\sigma\, dq\, dv.$

$e^{i(v+q^+)\cdot k}$ $\quad e^{iv\cdot k}$

Sketch: $\begin{cases} (v')^2 \le v^2 + v_*^2 \le 2R^2 \\ (v_*')^2 \le v^2 + v_*^2 \le 2R^2 , \end{cases}$ $\quad |q| \le 2R$



Restrict $f(v)$ on the cube $[-T,T]^3$ with $T \ge (3+\sqrt{2})R$.

$\begin{cases} f(v) = 0 \text{ on } [-T,T]^3 \setminus B_0(R) \\ f(v) \ne 0 \text{ on } B_0(R) \end{cases}$

Avoid Aliasing error.

Step3: Spectral projection of the Collision operator.

$$\begin{cases} f_N(v) = \sum_{k=-N}^{N} \boxed{\hat{f}_k} e^{ik\cdot v}, \quad k = (k_1,k_2,k_3) \in \mathbb{Z}^3 \\ \hat{f}_k = \frac{1}{(2\pi)^3}\int_{[-\pi,\pi]^3} f(v) e^{-ik\cdot v} dv, \end{cases} \quad \begin{cases} T = \pi \\ R = \lambda\pi, \ \lambda = \frac{3+\sqrt{2}}{2}. \end{cases}$$

$$\boxed{\hat{Q}_k} = \int_{[-\pi,\pi]^3} Q^R(f_N, f_N)\, e^{-ik\cdot v}\, dv, \quad k = -N, \cdots 0, \cdots N.$$

$$= \sum_{\substack{l,m=-N\\ l+m=k}}^{N} \hat f_l \hat f_m \hat\beta(l,m)$$

$$\boxed{\frac{\partial \hat f_k}{\partial t} = \hat Q_k = \sum \hat f_l \hat f_m \hat\beta(l,m)}$$

where the Boltzman kernel $\frac{1}{\sin\theta}$ $b(\cos\theta)|q|^\nu$ modes $\hat\beta(l,m) = \hat B(l,m) - \hat B(m,m)$.

$$\boxed{\hat B(l,m) = \int_{B_0(2\lambda\pi)} \int_{S^2} B(|q|,\cos\theta)\, e^{-i(l\cdot q^+ + m\cdot q^-)}\, d\sigma\, dq.}$$

Remark: (i) $\hat B(l,m) = \hat B(-l,m) = \overline{\hat B(l,-m)} = \hat B(l,-m)$.

(ii) They depend only on $|l-m|$, $|l+m|$, and the angle between $(l+m)$ and $(l-m)$

(iii) For some special collisional kernel, $\hat B(l,m)$ can be further simplified

VHS: $B(|q|,\sigma) = C_\nu |q|^\nu$

$$\boxed{\hat B(l,m) = \int_0^1 r^{2+\nu}\,\mathrm{sinc}(\xi r)\,\mathrm{sinc}(\eta r)\,dr := F_\nu(\xi,\eta)}$$

where $\xi = |l+m|\lambda\pi$, $\eta = |l-m|\lambda\pi$.

In particular,

$$\begin{cases} \nu = 0, \text{ Maxwell molecule. } & F_0(\xi,\eta) = \dfrac{p\sin(q) - q\sin(p)}{2\xi\eta pq} \\ \nu = 1, \text{ Hard-sphere, } & F_1(\xi,\eta) = \dfrac{q\sin(q)+\cos(q)}{2\xi\eta q^2} - \dfrac{p\sin(p)+\cos(p)}{2\xi\eta p^2} - \dfrac{2}{p^2q^2}. \end{cases}$$

where $p = (\xi+\eta)$, $q = (\xi-\eta)$.

$$\Rightarrow \hat Q_k = \sum_{m=-N}^{N} \hat f_{k-m} \hat f_m \hat\beta(k-m,m)$$

Step IV: Let Residual be L^2-orthogonal to element in $\mathbb{P}_N$

$$\int_{[-\pi,\pi]^3} \left(\frac{\partial f_N}{\partial t} - Q^R(f_N,f_N) \right) e^{-ik\cdot v}\, dv = 0. \quad k = -N, \cdots, N$$

to obtain the ODE system by the Fourier coefficients $\hat f_k$.

$$\Rightarrow \boxed{\frac{\partial \hat f_k}{\partial t} = \hat Q_k = \sum_{m=-N}^{N} \hat f_{k-m} \hat f_m \hat\beta(k-m,m), \quad k = -N, \cdots, N.}$$

Proposition: Let $f \in L^2([-\pi,\pi]^3)$, $\begin{pmatrix} \rho \\ \rho u \\ \rho e \end{pmatrix} := \int_{[-\pi,\pi]^3} f \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv$

Then we have the following properties for $\boxed{\text{"}f_N\text{"}}$

(i)

$$\begin{pmatrix} \rho_N \\ \rho u_N \\ \rho e_N \end{pmatrix} := \int_{[-\pi,\pi]^3} f_N \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv = \int_{[-\pi,\pi]} f \begin{pmatrix} 1 \\ v_N \\ (v^2)_N \end{pmatrix} dv$$

$v_N \to P_N v$, $(v^2)_N \to P_N v^2$.

$$= \int_{[-\pi,\pi]^3} f_N \begin{pmatrix} 1 \\ v_N \\ (v^2)_N \end{pmatrix} dv = (2\pi)^3 \sum_{k=-N}^{N} \hat{f}_k \begin{pmatrix} \delta_{k0} \\ \hat{v}_k \\ (\hat{v_2})_k \end{pmatrix}$$

(ii) $\boxed{\rho = \rho_N}$, $|\rho u - \rho u_N| \le \frac{C_1}{N^{\frac{1}{2}}} \|f\|_{L^2}$,

$|\rho e - \rho e_N| \le \frac{C_2}{N^{\frac{3}{2}}} \|f\|_{L^2}$.

Conserved Spectral Method.

Sketch of Proof: Mass conservation

$$\rho_N(t) = (2\pi)^3 \boxed{\sum_{k=-N}^{N} \hat{f}_k \delta_{k0}} = (2\pi)^3 \hat{f}_0 = \cancel{(2\pi)^3} \frac{1}{\cancel{(2\pi)^3}} \int_{[-\pi,\pi]^3} f e^{ik\cdot v} dv \Big|_{k=0}$$

$$= \int_{[-\pi,\pi]^3} f \, dv$$

$$= \rho(t)$$

$$\frac{\partial \int f \, 1 \, dv}{\partial t} = \int Q^R(f,f)\, 1 \, dv = 0$$

"1"

$\Rightarrow \rho(t) = \rho(0)$.