Lecture_2

Thursday, 27 January 2022 5:02 pm

$\{f^M(t)\} \in C^1([0,T]; L^1(\mathbb{R}^3))$ to Cauchy Problem $(2) = \begin{cases} \partial_t f^M = Q^M(f^M) \\ f^M(t=0,v) = f_0(v) \end{cases}$

Remove the Cutoff "M": $\xrightarrow{M \to \infty} \boxed{f}$ $\Downarrow$ $\begin{cases} \partial_t f = Q(f,f) \\ f(t=0,v) = f_0(v) \end{cases}$ $Q(f,f)$?

Compact argument for $\{f^M\}$, as $M \to \infty$.

Dunford-Pettis compact argument (Non-constrictive).

$\Leftrightarrow$ (weakly compact sets in L^1)

Let $\mathcal{F}$ be a $\boxed{\text{bounded}}$ set in $L^1(\Omega)$. Then $\mathcal{F}$ has compact closure in weak topology $\sigma(L^1, L^\infty)$ if and only if "$\boxed{\text{Equi-integrable}}$", that is

(a) $\begin{cases} \forall \varepsilon > 0, \exists \delta > 0, \text{ such that} \\ \int_{|A|} |f(v)| dv < \varepsilon, \ \forall A \subset \Omega, \text{measurable with } |A| < \delta \\ \text{for any } f \in \mathcal{F} \end{cases}$

and

(b) $\begin{cases} \forall \varepsilon > 0, \exists w \subset \Omega, \text{ measurable with } |w| < \infty \text{ such that} \\ \int_{\Omega \setminus w} |f| dv < \varepsilon, \ \text{for any } f \in \mathcal{F}. \end{cases}$

$\mathcal{F} = \{f^M(v)\}$ in $L^1(\mathbb{R}^3)$ ① $\underbrace{\|f^M\|_{L^1} = \int_{\mathbb{R}^3} f^M(v) dv = \int_{\mathbb{R}^3} f_0(v) dv = 1}_{\text{bounded.}}$

② By energy conservation:

$$\int_{|v|>R} f^M(t,v) dv \le \frac{1}{R^2} \underbrace{\int_{\mathbb{R}^3} |v|^2 f^M(t,v) dv}_{\substack{= \int_{\mathbb{R}^3} |v|^2 f_0(v) dv \\ := 2E_0 = 2E}} \le \frac{2E}{R^2} < \varepsilon$$

③ To check Condition (a), "A" $|A| < \delta$, s.t.

if A is a set such that $\boxed{\int_A dv \le \varepsilon}$, we have from the H-theorem and energy conservation, for $a>1$:

$$\int_A f^M(t,v)\,dv \le \frac{1}{\ln a}\int_A f^M(t,v)\ln f^M(t,v)\,dv\,\chi_{\{f^M(t)>a\}} + a\varepsilon$$

$$\le \frac{1}{\ln a}\int_{\mathbb{R}^3} f^M(t,v)\ln f^M(t,v)\,\chi_{\{f^M(t)>1\}} + a\varepsilon$$

$$\le \frac{1}{\ln a}H(f_0) - \frac{1}{\ln a}\int_{\mathbb{R}^3} f^M(t,v)\ln f^M(t,v)\,dv\,\chi_{\{f^M(t)<1\}} + a\varepsilon$$

$f^M(t)\le 1 \Rightarrow \begin{cases} e^{-v^2}<f^M<1 \\ f^M\le e^{-v^2}\end{cases}$

$$\le \frac{1}{\ln a}H(f_0) + a\varepsilon - \frac{1}{\ln a}\int_{\mathbb{R}^3} f^M(t,v)\ln f^M(t,v)\,dv\,\chi_{\{e^{-v^2}<f^M<1\}} - \frac{1}{\ln a}\int_{\mathbb{R}^3} f^M(t,v)\ln f^M(t,v)\,dv\,\chi_{\{f^M\le e^{-v^2}\}}$$

$x^{\frac12}|\ln x|$ bounded, $x<1$.

$$\le \frac{1}{\ln a}H(f_0) + a\varepsilon + \frac{1}{\ln a}\Big[\underbrace{\int_{\mathbb{R}^3} f^M(t,v)v^2\,dv}_{E} + \underbrace{C\int e^{-\frac{v^2}{2}}dv}_{=C'}\Big]$$

$\boxed{a=\varepsilon^{-\frac12}}$

$$\le -\frac{2H(f_0)}{\ln\varepsilon} + \sqrt{\varepsilon} - \frac{C(1+E)}{\ln\varepsilon}$$

Then we can apply the Dunford–Pettis Theorem to $\{f^M\}$ in $L^1([0,T]\times\mathbb{R}^3)$ to extract a weakly convergent subsequence with limit function $f\in L^1([0,T]\times\mathbb{R}^3)$.

$\Rightarrow \int_{\mathbb{R}^3}|v|^2 f(t,v)\,dv \le \int_{\mathbb{R}^3}|v|^2 f_0(v)\,dv$ for $t\in[0,T]$.

Due to the weak convergence; for any $\phi(t)$, $|\chi_H(v)|v|^2|$
$\phi(t)\in C([0,T])$, $\in L^\infty$

$$\int_0^T \phi(t)\int_{\mathbb{R}^3}\chi_H(v)|v|^2 f^n(v)\,dv\,dt \to \int_0^T\phi(t)\int_{\mathbb{R}^3}\chi_H(v)|v|^2 f(v)\,dv\,dt.$$

where $\chi_H(r) = \begin{cases} r, & \text{if } |r|\le H \\ 0, & \text{otherwise.}\end{cases}$

$$\Rightarrow \frac12\int_0^T\phi(t)\int_{\mathbb{R}^3}|v|^2 f(v)\,dv\,dt \le \int_0^T\phi(t)\,\underbrace{E}_{\frac12\int_{\mathbb{R}^3}f_0(v)|v|^2dv}$$

To prove that $\boxed{f}$ is a solution to the $\{\partial_t f = Q(f,f)$

We have to check the convergence of the ($f(t=0,v)=f_0(v)$)
collision operator:

$$Q^n(f^n,f^n):=Q^{Mn}(f^n,f^n)\longrightarrow Q(f,f).$$

to do it, we define the family of norms:

$$\|f\|_{L^1_s}=\int_{\mathbb{R}^3}\langle v\rangle^s|f(v)|\,dv.=\int_{\mathbb{R}^3}(1+v^2)^{\frac{s}{2}}|f(v)|\,dv.$$

the associated family of Banach space:

$$L^1_s=\{f(v);\ \|f\|_{L^1_s}<\infty\}.$$

Lemma (Rough Povzner Inequality) Suppose $s\geq 2$, for $f,g\in L^1_s$
$f\geq 0$, $g\geq 0$. Then the following inequality holds,

$$\underbrace{\int_{\mathbb{R}^3}(1+|v|^2)^{\frac{s}{2}}|Q(f,g)|\,dv}\leq \underline{C_s\left[\|f\|_{L^1_s}\|g\|_{L^1_2}+\|g\|_{L^1_s}\|f\|_{L^1_2}\right]}$$

$$\Downarrow$$

$$\int_{\mathbb{R}^3}\langle v\rangle^s|Q(f,g)|\,dv$$

$$\Downarrow$$

$$\underline{\|Q(f,g)\|_{L^1_s}}\qquad \Leftrightarrow Q^m \qquad \begin{matrix}\|Q^+(g,f)\|_{L^1_s}\\ \|Q^-(g,f)\|_{L^1_2}\end{matrix}\qquad \text{"s+1" of } f$$

Pf: For both a,b are positive number, $b>a$, and any $v\in[0,1]$

$$(a+b)^s\leq a^s+b^s+2^{s-1}b^{s-1}a\leq a^s+b^s+2^{s-1}b^{s-v}a^v.$$

hence, for positive a and b, $\Downarrow$

$$\boxed{a^s+b^s\leq(a+b)^s\leq a^s+b^s+C_s(b^{s-v}a^v+a^{s-v}b^v)}$$

$$\Rightarrow\qquad (1+|v|^2)+(1+|v_*|^2)\ \xlongequal{\text{Energy conservation}}\ (1+|v'|^2)+(1+|v_*'|^2)$$

$$\Rightarrow(1+|v'|^2)^{\frac{s}{2}}+(1+|v_*'|^2)^{\frac{s}{2}}\leq(\underline{1+|v'|^2+1+|v_*'|^2})^{\frac{s}{2}}=(\underline{1+|v|^2+1+|v_*|^2})^{\frac{s}{2}}$$

$$\leq(1+|v|^2)^{\frac{s}{2}}+(1+|v_*|)^{\frac{s}{2}}$$

$$+C_s\Big[(1+|v|^2)^{\frac{v}{2}}(1+|v_*|^2)^{\frac{s-v}{2}}$$

$$+(1+|v|^2)^{\frac{s-v}{2}}(1+|v_*|^2)^{\frac{v}{2}}\Big]$$

$$\Rightarrow \langle v'\rangle^s + \langle v_*'\rangle^s - \langle v\rangle^s - \langle v_*\rangle^s$$

$$\leq C_s \left[\underbrace{\langle v\rangle^{\nu} \langle v_*\rangle^{s-\nu} + \langle v\rangle^{s-\nu}\langle v_*\rangle^{\nu}} \right].$$

Now, take $\phi(v) = \langle v\rangle^s$ into weak formulation

$$= \|Q(g,f)\|_{L^1_s}$$

$$= \frac{1}{2}\int_{S^2} |(v-v_*)w| \left[\phi' + \phi_*' - \phi - \phi_*\right] f(v) g(v_*)\, dv\, dv_*. \checkmark$$

$$= \frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3}\int_{S^2} \underbrace{|(v-v_*)\cdot w|}_{=|v-v_*|} \left[\langle v'\rangle^s + \langle v_*'\rangle^s - \langle v\rangle^s - \langle v_*\rangle^s\right] f(v) g(v_*)\, \underset{dw}{\widehat{d}} v\, dv_*$$

$$\leq \underbrace{(1+|v|^2)^{\frac{1}{2}}(1+|v_*|^2)^{\frac{1}{2}}}_{\langle v\rangle \cdot \langle v_*\rangle}$$

$$\overset{\nu=1}{\Longrightarrow} \leq C_s \frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3} \langle v\rangle \cdot \langle v_*\rangle \left[\langle v\rangle \langle v_*\rangle^{s-1} + \langle v_*\rangle\langle v\rangle^{s-1}\right] f(v)\, g(v_*)\, dv\, dv_*.$$

$$\leq C_s \frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3}\left[\langle v\rangle^2 \langle v_*\rangle^s + \langle v\rangle^s\langle v_*\rangle^2\right] f(v)\, g(v_*)\, dv\, dv_*$$

$$\leq C_s \left(\|f\|_{L^1_2}\|g\|_{L^1_s} + \|f\|_{L^1_s}\|g\|_{L^1_2}\right). \quad \#$$

$$f^M(t,v) = f_0(v) + \int_0^t Q(f^M, f^M)(\tau, v)\, d\tau$$

$$\Rightarrow \boxed{\|f^M(t)\|_{L^1_s}} \leq \|f_0\|_{L^1_s} + C_s\int_0^t \|Q(f^M, f^M)\|_{L^1_s}\, d\tau$$

$$\leq \|f_0\|_{L^1_s} + C_s\int_0^t \boxed{\|f^M(\tau)\|_{L^1_s}}\, \|f^M(\tau)\|_{L^1_2}\, d\tau$$

$$\leq 2E := \int |v|^2 f_0(v)\, dv$$

Gronwall's inequality

$$\Longrightarrow \boxed{\|f^M(t)\|_{L^1_s} \leq e^{2C_sEt}\, \|f(0)\|_{L^1_s}} \checkmark$$

$$Q^-(f,f) = f\, \mathcal{L}^M[f] = f\int_{\mathbb{R}^3}\int_{S^2} |v-v_*|\, f(v_*)\, dw\, dv_*$$

By the inequality,

$$\mathcal{L}^M[f] \leq C\int_{\mathbb{R}^3} |v-v_*|\, f(v_*)\, dv_*.$$

$$\leq C \int_{\mathbb{R}^3} (|v| + |v_*|) f(v_*)\, dv_*$$

$$\leq C (|v| \, \|f\|_{L^1} + \|f\|_{L^1_2})$$

then, $\frac{d}{dt} \|f\|_{L^1_2} = \|Q_+^M(f,f)\|_{L^1_2} - \|f L^M[f]\|_{L^1_2}$

$$= 0$$

$$\Rightarrow \|Q_+^M(f,f)\|_{L^1_2} = \|f L^M[f]\|_{L^1_2}$$

$$\leq C \int_{\mathbb{R}^3} \langle v\rangle^2 (|v| \, \|f\|_{L^1} + \|f\|_{L^1_2}) f(v)\, dv$$

$$\leq C [\|f\|_{L^1_3} \|f\|_{L^1} + \|f\|^2_{L^1_2}]$$

So, $f^M(t,v) - f^M(s,v) = \int_s^t Q^M(f^M, f^M)(\tau, v)\, d\tau$

$$\Rightarrow \|f^M(t,\cdot) - f^M(s,\cdot)\|_{L^1_2} \leq \int_s^t \|Q^M(f^M, f^M)\|_{L^1_2}(\tau,\cdot)\, d\tau$$

$$\leq C(T)\, |t-s|$$

$$\Rightarrow f \in C([0,T]; L^1_2).$$

we can extract a subsequence, denoted by $f^n(t)$, which converges weakly in $L^1(\mathbb{R}^3)$ to $f(t)$, for all $t \in [0,T]$.

$\Rightarrow$ for $t \in [0,T]$, $\phi_{(v,v_*)} \in L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} \phi(v,v_*) f^n(v) f^n(v_*)\, dv\, dv_* \longrightarrow \int \phi(v,v_*) f(v) f(v_*)\, dv\, dv_*$$

$$\Rightarrow \left| \int_{\mathbb{R}^3} [Q_+^n(f^n) - Q_+(f)](v)\, \phi(v)\, dv \right| \longrightarrow 0 \quad v', v_*'$$

(I) $$\leq \left| \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{S^2} |(v-v_*)\cdot w| \, \chi_M(|v-v_*|)\, \phi(v') [\underbrace{f^n(v) f^n(v_*) - f(v) f(v_*)}_{\xrightarrow{n\to\infty} 0}]\, d\sigma\, dv\, dv_* \right|$$

(II) $$+ C \|\phi\|_{L^\infty} \left(\int_{\mathbb{R}^3 \times \mathbb{R}^3} \left| \int_{|v|+|v_*| \geq \frac{M}{2}} \underbrace{(|v| + |v_*|) f^n(v) f^n(v_*)}_{\xrightarrow{M\to\infty} 0} \right| dv\, dv_* \right)$$

(III) $$+ C \|\phi\|_{L^\infty} \left(\int_{\mathbb{R}^3 \times \mathbb{R}^3} \left| \int_{|v| > \frac{M}{2}} (|v| + |v_*|) f(v) f(v_*) \right| dv\, dv_* \right).$$

$M \to \infty$

Exercise 5: Check the weak convergence of the loss term

$$Q^-(f,f) = f\mathcal{L}[f].$$

Theorem: Let $f_0 \geq 0$ be an initial datum with finite entropy and such that $\|f_0\|_{L^1_3}$. Then, there exist $f \in C([0,T]; L^1_2)$ satisfying:

$$f(t,v) = f_0(v) + \int_0^t Q(f(\tau,v), f(\tau,v))\, d\tau.$$

where, $f(t) \in L^1_3$ and $\|f(t)\|_{L^1_s} \leq e^{2C_sEt}\|f_0\|_{L^1_s}$, $s \geq 2$.

For mass and energy, we have $\|f(t,\cdot)\|_{L^1} = 1$ and $\frac{1}{2}\int_{\mathbb{R}^3} |v|^2 f(v)\,dv = E$.

#

Theorem: (Uniqueness and some regularity property)

Let $f_0 \geq 0$ be initial datum with finite entropy and $f_0 \in L^1_4$.

Then, there exist a $\boxed{\text{unique}}$ $f \in C^1([0,T]; L^1)$ satisfying

$$\begin{cases} \partial_t f = Q(f,f) \\ f(t=0, v) = f_0(v) \end{cases}$$

Where $f(t) \in L^1_4$, and H-theorem holds, such that

$$H(f(t)) \leq H(f_0)$$

Pf:
$$\begin{cases} \|Q_+(f,g)\|_{L^1_s} \leq C_s\left[\|f\|_{L^1_{s+1}}\|g\|_{L^1} + \|g\|_{L^1_{s+1}}\|f\|_{L^1}\right] \\ \|f\mathcal{L}[g] + g\mathcal{L}[f]\|_{L^1_s} \leq C_s\left[\|f\|_{L^1_{s+1}}\|g\|_{L^1} + \|g\|_{L^1_{s+1}}\|f\|_{L^1}\right] \end{cases}$$

if $f_0 \in L^1_{s+1}$,

$$\|f(t) - f(s)\|_{L^1_s} \leq \int_s^t \|Q(f,f)(\tau,\cdot)\|_{L^1_s}\, d\tau$$
$$\leq C_{s,T}|t-s|$$

$f(t) \in C^1([0,T], L^1_2)$

$$\Rightarrow \left\| \frac{f(t+h) - f(t)}{h} - Q(f(t), f(t)) \right\|_{L^1_2}$$
$$\leq \frac{1}{h}\int_t^{t+h} \|Q(f(\tau) + f(t), f(\tau) - f(t))\|_{L^1_2}\, d\tau$$

$\tau \in (t, t+h)$

$$\leq C h^{-1} \int_t^{t+h} \underbrace{\|f(\tau)+f(t)\|_{L^1_3}}_{\text{"}\sim h\text{"}} \underbrace{\|f(\tau)-f(t)\|_{L^1_3}}_{\text{"}\sim h\text{"}} \, d\tau$$

$$\leq C' h.$$

For the $\boxed{\text{uniqueness}}$, we have to suppose that f, g are two solutions with respect to the same initial datum.

$$\begin{cases} \frac{\partial}{\partial t} f(t,v) = Q(f,f)(t,v) \\ f(t=0,v) = f_0(v) \end{cases} - \begin{cases} \frac{\partial}{\partial t} g(t,v) = Q(g,g)(t,v) \\ g(t=0,v) = f_0(v) \end{cases}$$

$$\Rightarrow \begin{cases} \frac{\partial}{\partial t} [f(t,v) - g(t,v)] = \boxed{Q(f,f) - Q(g,g)} = Q(f-g, f+g) \\ (f-g)(t=0,v) = 0. \end{cases}$$

$$\xrightarrow{\|\cdot\|_{L^1_2}} \frac{d}{dt} \|f(t,\cdot) - g(t,\cdot)\|_{L^1_2} = \int_{\mathbb{R}^3} \langle v \rangle^2 \, \mathrm{sgn}[f(t) - g(t)] \, Q(f-g, f+g) \, dv.$$

$$\xrightarrow{R\text{-}H\text{-}S} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} |(v-v_*)\cdot w| (1+|v|^2) \, \mathrm{sgn}(f-g)$$
$$\{ (f+g)(v_*')(f-g)(v') + (f+g)(v')(f-g)(v_*')$$
$$- (f+g)(v_*)(f-g)(v) - (f+g)(v)(f-g)(v_*) \} \, dv \, dv_* \, dw$$

$\boxed{\mathrm{sgn}\, x \cdot x = |x|}$

$$\leq \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} |(v-v_*)\cdot w| (1+|v|^2) \{ (f+g)(v_*') |f-g|(v') + (f+g)(v')$$
$$|f-g|(v_*') - (f+g)(v_*) |f-g|(v)$$
$$+ (f+g)(v)(f-g)(v_*) \} \, dv \, dv_*$$

$$\overbrace{\leq C \|f(t) - g(t)\|_{L^1_2}}^{\|Q(f,g)\|_{L^1_2}} \underbrace{\|f(t) + g(t)\|_{L^1_3}}_{\leq C'}$$

$$\Rightarrow \frac{d}{dt} \|f(t) - g(t)\|_{L^1_2} \leq C' \|f(t) - g(t)\|_{L^1_2}$$

$$\Rightarrow \|f(t) - g(t)\|_{L^1_2} \leq \underbrace{e^{C't}}_{0 \leq t \leq T} \|f_0 - g_0\|_{L^1_2}$$

$\Rightarrow \|f(t) - g(t)\| \ldots$

$\rightarrow \|f(T)-g(T)\|_{L^1_2} \le C(T)\|f_0-g_0\|_{L^1_2}$ #

Since $\{f^n\} \rightarrow f$ weakly, and $H(f)=\int f\log f$ convex function,

$H(f) \le \liminf_{n\to\infty} H(f_n)$.

$\{f\}$ in L^1-theory.

L^∞-Theory:

Theorem: Suppose $f(v) \le \frac{C}{(1+|v|^2)^{\frac{s}{2}}} := \frac{C}{\langle v\rangle^s}$ with $s>6$. Then the solution to homogeneous Boltzmann $f(t,v)$ satisfies:

$$\sup_{t>0} \|f(t,\cdot)\|_{L^\infty} < C.$$

where C depends only on f_0. In addition, for almost all $v \in \mathbb{R}^3$ and all $t>0$, $f(t,v)$ is differentiable in time and $\frac{\partial f(t,v)}{\partial t} = Q(f,f)(t,v)$ point-wise.

Pf: See "The Mathematical Theory of Dilute Gases"

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