Lecture_1

Wednesday, 26 January 2022 3:30 pm

Properties and estimate of Collision Operator

- Bilinear: $Q(g,f) = Q^+(g,f) - Q^-(g,f)$

No frontal collision should occur: Assumption (I)

$\exists \theta_b > 0$, $\operatorname{supp} b(\cos\theta) \subset \{\theta \mid 0 \le \theta \le \pi - \theta_b\}$ ✓

- Quadratic form: $Q(f,f) = Q^+(f,f) - Q^-(f,f)$

$b(\cos\theta)\big|_{0\le\theta\le\pi} \iff [b(\cos\theta) + b(\cos(\pi-\theta))]\,\mathbb{1}_{0\le\theta\le\frac{\pi}{2}}$

symmetric between g, f $\quad \|Q^+(\overset{v_*}{g},\overset{v}{f})\| \le \cdots$

$\Updownarrow$ Assumption (II)

$\|Q^+(f,g)\| \le \cdots$

$\exists \theta_b > 0$, $\operatorname{supp} b(\cos\theta) \subset \{\theta \mid \theta_b \le \theta \le \pi\}$

$Q^-(f,f) = f\,\mathcal{L}[f] = f(v) \underbrace{\int_{\mathbb{R}^3}\int_{S^2} B(v-v_*,\sigma) f(v_*)\, d\sigma\, dv_*}_{\mathcal{L}[f]}$

Theorem: Let $k, \eta \in \mathbb{R}$, $s \in \mathbb{R}_+$, $p \in [1,\infty]$, and let B be $B = b(\cos\theta)\,\Phi(|v-v_*|)$, satisfying the assumption (I). Then, the following estimate hold:

$$\|Q^+(g,f)\|_{L^p_\eta(\mathbb{R}^d)} \le C_{k,\eta,p}(B)\, \|g\|_{L^1_{|k+\eta|+|\eta|}(\mathbb{R}^d)} \|f\|_{L^p_{k+\eta}(\mathbb{R}^d)} \quad (*)$$

$$\|Q^+(g,f)\|_{W^{s,p}_\eta(\mathbb{R}^d)} \le C_{k,\eta,p}(B)\, \|g\|_{W^{\lceil s\rceil,1}_{|k+\eta|+|\eta|}(\mathbb{R}^d)} \|f\|_{W^{s,p}_{k+\eta}(\mathbb{R}^d)} \quad (**)$$

$|\nabla Q^+(g,f)|$

where $C_{k,\eta,p}(B) = C\left(\sin\left(\frac{\theta_b}{2}\right)\right)^{\min(\eta,0)-\frac{2}{p'}} \|b\|_{L^1(S^{d-1})} \|\Phi\|_{L^\infty_{-k}}$.

If Assumption (I) is replaced by Assumption (II), the same estimate hold with $Q^+(g,f)$ replaced by $Q^+(f,g)$.

Pf: By duality,

$$\|Q^+(g,f)\|_{L^p_\eta} := \sup_{\|\phi\|_{L^{p'}_{-\eta}}\le 1}\left\{\int_{\mathbb{R}^d} Q^+(g,f)(v)\,\phi(v)\,dv\right\}.$$

step I: Pre-Post collisional velocity transformation.

$$(v, v_*, \sigma) \longmapsto \left(v', v_*', \frac{v-v_*}{|v-v_*|}\right), \quad |J| = 1$$

$$\sigma = \frac{v'-v_*'}{|v'-v_*'|}$$

$$\int_{\mathbb{R}^d} Q^+(g,f)\,\phi(v)\,dv = \int_{\mathbb{R}^d\times\mathbb{R}^d}\left(\int_{S^{d-1}} B(|v-v_*|,\sigma)\,\phi(v')\,d\sigma\right) g_* f\,dv_*\,dv.$$

for all $\|\phi\|_{L^{p'}_{-\eta}} \le 1$.

$\tau_{v_*} S(\tau_{-v_*}\phi)(v)$

Furthermore, we define linear operator S by

$$\boxed{S\phi(v) = \int_{S^{d-1}} B(|v|,\sigma)\,\phi\left(\frac{v+|v|\sigma}{2}\right)d\sigma}$$

$$B(|v|,\sigma) = b\cdot\Phi$$

Then,

$$\int_{\mathbb{R}^d} Q^+(g,f)\,\phi(v)\,dv = \int_{\mathbb{R}^d} g(v_*)\left(\int_{\mathbb{R}^d} f(v)\left[\tau_{v_*} S(\tau_{-v_*}\phi)\right](v)\,dv\right)dv_*.$$

step 2: Estimate for the operator S in weighted $L^1 - L^\infty$.

step 2.1: For $\|S\phi\|_{L^\infty}$, denote $v^+ = \left(\frac{v+|v|\sigma}{2}\right)$.

$$\boxed{\sin\left(\frac{\theta_b}{2}\right)|v| \le |v^+| \le |v|}$$

$$|v^+| = \left|\frac{v+|v|\sigma}{2}\right| \le \frac{|v|+|v|}{2} = |v|$$

$$\|S\phi\|_{L^\infty_{-k-\eta}} \le C\left(\sin\left(\frac{\theta_b}{2}\right)\right)^{\min(\eta,0)}\|\Phi\|_{L^\infty_{-k}}\|\phi\|_{L^\infty_{-\eta}}\|b\|_{L^1(S^{d-1})}$$

$\phi(v^+)$

step 2.2: For $\|S\phi\|_{L^1}$

$$\|S\phi\|_{L^1_{-k-\eta}} = \int_{\mathbb{R}^d}\int_{S^{d-1}} b(\cos\theta)\,\Phi(|v|)\,\langle v\rangle^{-k-\eta}\,|\phi(v^+)|\,d\sigma\,dv$$

$\|\cdot\|_{L^\infty}$ $\quad v^+$

$$\le \left(\sin\left(\frac{\theta_b}{2}\right)\right)^{\min(\eta,0)} \|\Phi\|_{L^\infty_{-k}}$$

$$\underbrace{\int_{\mathbb{R}^d}\int_{S^{d-1}} b(\cos\theta)|\phi(v^+)|\langle v^+\rangle^{-\eta}\, d\sigma\, dv}_{\Updownarrow\, v^+}$$

Take the transformation $v \xrightarrow[J]{} v^+ = \frac{v+|v|\sigma}{2}$

$\boxed{\text{Excise 2: Calculate the Jacobian } |J| = \frac{2^{d-1}}{\cos^2\frac{\theta}{2}}}$

Thank to the Assumption (I).

$v \to v^+$ always works out.

$\theta \to \pi$, $\cos^2\frac{\theta}{2} \to 0$.

$$\|S\phi\|_{L^1_{-k-\eta}} \le C\left(\sin\frac{\theta_b}{2}\right)^{\min(\eta,0)} \|\Phi\|_{L^\infty_{-k}}.$$

$$\int_{\mathbb{R}^d}\int_{S^{d-1}} b(\cos\theta)\underbrace{|\phi(v^+)|\langle v^+\rangle^{-\eta}}_{\Downarrow\, \|\phi\|_{L^1_{-\eta}}} \frac{2^{d-1}}{\cos^2\frac{\theta}{2}}\, dv^+ d\sigma$$

$$\le C\left(\sin\frac{\theta_b}{2}\right)^{\min(\eta,0)-2} \|\Phi\|_{L^\infty_{-k}} \|\phi\|_{L^1_{-\eta}} \|b\|_{L^1(S^{d-1})}.$$

Step 2.3: How to make it for $\|S\phi\|_{L^p}$

$\boxed{\text{Riesz – Thorin Interpolation}}$ $\quad \|S\phi\|_{L^1}, \|S\phi\|_{L^\infty}$

There exist
$$\begin{cases} \frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, & \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2} \\ k = \theta k_1 + (1-\theta)k_2, & l = \theta l_1 + (1-\theta) l_2. \end{cases}$$

if
$$\begin{cases} \|Tf\|_{L^{q_1}_{l_1}} \le C\|f\|_{L^{p_1}_{k_1}}, & q_1 = \infty, \ p_1 = \infty \\ \|Tf\|_{L^{q_2}_{l_2}} \le C\|f\|_{L^{p_2}_{k_2}} & q_2 = 1, \ p_2 = 1. \end{cases}$$

$$\Rightarrow \|Tf\|_{L^q_l} \le C\|f\|_{L^p_k}$$

$\boxed{\begin{array}{l}\text{For } p \in [1,\infty], \text{ by interpolation:} \\ \|S\phi\|_{L^{p'}_{-k-\eta}} \le C_{k,\eta,p'}(B)\, \|\phi\|_{L^p_{-\eta}}.\end{array}}$

step 3: $\|Q^+(g,f)\|_{L^p_\eta} := \left|\int_{\mathbb{R}^d} Q^+(g,f)(v)\phi(v)\,dv\right|$

$\Big|\underbrace{\int Q^+(g,f)\phi\,dv}_{\text{if } \|\phi\|_{L^{p'}_{-\eta}}\le 1}\Big| \le \|f\|\cdot\|g\|\cdot\|\phi\|_{L^{p'}_{-\eta}} \le 1$

$$\le \int_{\mathbb{R}^d} |g_*| \left(\int_{\mathbb{R}^d} |f| \left[\tau_{-v_*} S(\tau_{v_*}\phi)\right](v)\,dv\right) dv_*$$

$$\le \int_{\mathbb{R}^d} |g_*|\, \|f\|_{L^p_{k+\eta}}\, \|\tau_{-v_*} S(\tau_{v_*}\phi)\|_{L^{p'}_{-k-\eta}}\, dv_*$$

$v \mapsto v - v_*$

$$\le \|f\|_{L^p_{k+\eta}} \int_{\mathbb{R}^d} |g_*| \langle v_*\rangle^{|k+\eta|} \underbrace{\|S(\tau_{v_*}\phi)\|_{L^{p'}_{-k-\eta}}}_{\text{step 2}} dv_*$$

$$\le C_{k,\eta,p}(B)\|f\|_{L^p_{k+\eta}} \int_{\mathbb{R}^d} |g_*| \langle v_*\rangle^{|k+\eta|} \|\tau_{v_*}\phi\|_{L^p_{-\eta}}\, dv_*$$

$v \mapsto v + v_*$

$$\le C_{k,\eta,p}(B)\|f\|_{L^p_{k+\eta}} \underbrace{\|\phi\|_{L^{p'}_{-\eta}}}_{\le 1} \int_{\mathbb{R}^d} |g_*| \langle v_*\rangle^{|k+\eta|+|\eta|}\, dv_*$$

$$\le C_{k,\eta,p}(B)\|f\|_{L^p_{k+\eta}} \int_{\mathbb{R}^d} |g_*| \langle v_*\rangle^{|k+\eta|+|\eta|}\, dv_*$$

$$\le C_{k,\eta,p}(B)\|g\|_{L^1_{|k+\eta|+|\eta|}} \|f\|_{L^p_{k+\eta}}. \quad \#$$

This concludes estimate $(*)$.

For the estimate of $(**)$ $\qquad \partial(fg) = (\partial f)g + f(\partial g)$

$$\boxed{\nabla Q^\pm(g,f)(v) = Q^\pm(\nabla g, f) + Q^\pm(g, \nabla f)} \qquad \boxed{\partial^\nu(fg) = \sum_{\mu\le\nu}\binom{\nu}{\mu}\partial^\mu g\,\partial^{\nu-\mu} f}$$

Exercise 3: $\boxed{\text{hint:}}$ bilinearly and Galilean invariance of $Q(g,f)$

multi-index $\qquad \tau_h Q(g,f) = Q(\tau_h g, \tau_h f)$

For $s \in \mathbb{N}$, $\partial^\nu Q^+(g,f) = \sum_{\mu\le\nu}\binom{\nu}{\mu} Q^+(\partial^\mu g, \partial^{\nu-\mu} f)$.

$$\|Q^+(g,f)\|^p_{W^{s,p}_\eta} = \sum_{|\nu|\le s} \|\partial^\nu Q^+(g,f)\|^p_{L^p_\eta}$$

leibniz

$$\overset{\text{Leibniz}}{\underset{\text{rule}}{=}} \sum_{|\nu|\le s} \sum_{\mu\le\nu} \binom{\nu}{\mu} \underbrace{\| Q^+(\underline{\partial^\mu g}, \partial^{\nu-\mu} f)\|^p_{L^p_\eta}}$$

$$\underset{(*)}{\le} C_{k,\eta,p}(B) \sum_{|\nu|\le s} \sum_{\mu\le\nu} \binom{\nu}{\mu} \|\underline{\underline{\partial^\mu g}}\|^p_{L^1_{|k+\eta|+|\eta|}} \|\partial^{\nu-\mu} f\|^p_{L^p_{k+\eta}}$$

$$\le C_{k,\eta,p}(B) \|g\|^p_{W^{s,1}_{|k+\eta|+|\eta|}} \|f\|^p_{W^{s,p}_{k+\eta}}$$

This concludes the estimate of (**).

$$\underbrace{\|Q^-(f,f)\|_{L^p}} = \|\underline{f}\, \underset{\Downarrow}{L[f]}\|_{L^p}$$

$$\left\| \int b\, d\sigma \left[\underline{|v-v_*|^{\tilde\gamma}} * f \right] \right\|_{L^p} \xRightarrow{\text{Young inequality}}$$

Application: $\|Q(f,f)\|_{L^1} \le C \|f\|^2_{L^1}$ from $(*)$

1. A Existence Theorem for (Modified) homogeneous Boltzmann.

$$① \begin{cases} \partial_t f(t,v) = Q^M(f,f)(t,v) \\ f(t=0,v) = f_0(v) \end{cases}$$

(Modified Cutoff Hard-sphere) Collision Operator:

$$Q^M(\underline{g},\underline{f}) = \frac{1}{2} \int_{\mathbb{R}^3} \int_{S^2} |(v-v_*)\cdot \omega|\, \underline{\chi_M(|v-v_*|)} [g_*' f' + g' f_*' - g_* f - f_* g]\, d\omega\, dv_*$$

where $\chi_M : \mathbb{R}_+ \longmapsto \mathbb{R}$.

$$\chi_M(\underline{r}) := \begin{cases} 1, & \text{if } r \le M \\ 0, & \text{otherwise.} \end{cases}$$

$$① \to \boxed{②} \Rightarrow \begin{cases} \partial_t f^M(t,v) = Q^M(f^M, f^M)(t,v) \\ f^M(t=0,v) = f_0(v) \in L^1(\mathbb{R}^3) \end{cases}$$

$$\begin{cases} \|Q^M(f,f) - Q^M(g,g)\|_{L^1} = \|Q^M(f+g, f-g)\|_{L^1} \quad \checkmark \\ \qquad \le CM \|f+g\|_{L^1} \cdot \|f-g\|_{L^1} \\ \|Q^M(f,f)\|_{L^1} \le CM \|f\|^2_{L^1} \end{cases}$$

Local existence of f^M:

$$f^M(t,v) = \underbrace{f_0(v) + \int_0^t Q^M(f^M, f^M)(\tau, v)\, d\tau}_{:= P[f^M](t,v)}.$$

By Banach fixed point Theorem: $X := \{ f \in C^1([0,T]; L^1(\mathbb{R}^3))$ i.e. $\|f(\tau,\cdot)\|_{L^1} \le C \}$

Step I: For any $f_0(v) \in L^1$, $f^M \in X$,
then $P[f^M] \in L^1$, for $t \in [0,T]$

$$\|P[f^M]\|_{L^1(\mathbb{R}^3)} = \left\| f_0 + \int_0^T Q^M(f^M, f^M)(\tau, v)\, d\tau \right\|_{L^1(\mathbb{R}^3)}$$

$$\le \|f_0\|_{L^1(\mathbb{R}^3)} + \int_0^t \|Q^M(f^M, f^M)\|_{L^1}(\tau)\, d\tau$$

$$\le \underbrace{\|f_0\|_{L^1(\mathbb{R}^3)}}_{<\infty} + CM \underbrace{\|f^M\|_{L^1}^2}_{<\infty} T$$

$$\le \infty$$

Step II: For both $f^M, g^M \in X$ with the same initial datum

$$\|P[f^M] - P[g^M]\|_{L^1(\mathbb{R}^3)} = \left\| \int_0^T Q^M(f^M, f^M) - Q^M(g^M, g^M)\, d\tau \right\|_{L^1}$$

$$= \int_0^T \|Q^M(f^M, f^M) - Q^M(g^M, g^M)\|_{L^1}(\tau)\, d\tau$$

$$\le \int_0^T CM \|f^M + g^M\|_{L^1} \|f^M - g^M\|_{L^1}\, d\tau$$

$$\le \underbrace{CM(\|f^M\|_{L^1} + \|g^M\|_{L^1})T}_{<1} \|f^M - g^M\|_{L^1}$$

$\|f_0\| \in L^1$

Combined step I and II,
$\exists f^M \in C^1([0,T]; L^1(\mathbb{R}^3))$ for a sufficiently small $T > 0$.

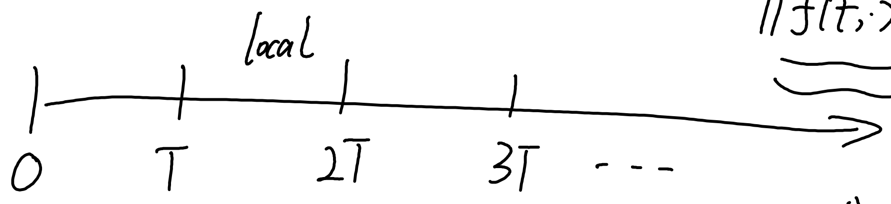
#

Assume the initial datum positive and normalized, i.e.

$$\boxed{\int_{\mathbb{R}^3} f_0(v)\,dv = 1} \xrightarrow{\text{mass conservation}} \int_{\mathbb{R}^3} f(t,v)\,dv = 1.$$

$$\Big\Downarrow \text{ positivity of } f$$

$$\|f(t,\cdot)\|_{L^1} \overset{?}{=} 1.$$



remove "M" $\to$ "f"

$\|f\|_{L^1}$

$\|f_0\|_{L^1} = \|f(T)\|_{L^1} = \|f(2T)\|_{L^1}$

Prove the Positivity of $f^M \Leftrightarrow g$

$$② \Longleftrightarrow ③ \begin{cases} \partial_t g + \mu g = P^M[g](t,v) \\ g(t=0,v) = f_0(v). \end{cases}$$

where $P^M[g](t,v) = Q^M(g,g)(t,v) + \mu g \int_{\mathbb{R}^3} g(v)\,dv$

$\mu > 0$ will be chose sufficiently large later.

By the local existence, we've already obatined a local solution f^M to ③

i.e. $\Longrightarrow$ if we prove g in ③ is positive, then so is f^M.

Because $P^M[g]$ is Lipschitz continuous, ③ has unique solution

$$\text{iteration} \begin{cases} g^n = e^{-\mu t} f_0 + \int_0^t e^{-\mu(t-\tau)} P^M[g^{n-1}]\,d\tau \\ g^0(v) = 0 \end{cases} \quad (\star).$$

Step I: Prove P^M is positive monotone operator in the sense

$$\boxed{P^M[f] \geq P^M[g], \text{ if } f \geq g \geq 0}$$

Recall that ...

Recall that $\begin{cases} Q_+^M(g,f) = \frac{1}{2}\int_{\mathbb{R}^3}\int_{S^2} |(v-v_*)\cdot\omega|\,\chi_M(|v-v_*|)\,[g'_* f' + f'_* g']\,d\omega\, dv_* \\ Q_-^M(f,f) = \int_{\mathbb{R}^3}\int_{S^2} |(v-v_*)\cdot\omega|\,\chi_M(|v-v_*|)\,[f f_*]\,d\omega\, dv_* \\ \qquad\quad = f\, L^M[f]. \end{cases}$

By using symmetry property:

$$\underline{P^M[f] - P^M[g]} = \underline{Q_+^M(f+g, f-g)} - \frac{1}{2}\{(f+g)L^M[f-g] + (f-g)L^M[f+g]\}$$
$$+ \frac{1}{2}\mu\{(f+g)\int_{\mathbb{R}^3}(f-g)\,dv + (f-g)\int_{\mathbb{R}^3}(f+g)\,dv\}.$$

≥ 0, if $f-g>0$.

$$L^M[f] = \int_{\mathbb{R}^3}\int_{S^2} |\omega\cdot(v-v_*)|\,\chi(|v-v_*|)\, f(v_*)\, d\omega\, dv_*$$
$$\leq CM \int_{\mathbb{R}^3} f(v_*)\, dv_*.$$

$$\Leftarrow L^M[f-g] \leq \boxed{CM} \int (f-g)(v_*)\,dv_*.$$

So the monotonicity holds for $P^M[f]$, if we select μ big enough, e.g., $\mu > CM$.

$\Longrightarrow$ $g^n \geq g^{n-1} \geq \cdots$ is a monotonically increasing sequence.

Step II: Integration (⊗) with respect to v.

$$\underbrace{\int_{\mathbb{R}^3} g^n(v)}_{\leq 1} = e^{-\mu t}\underbrace{\int f_0\, dv}_{=1} + \int_0^t e^{-\mu(t-\tau)} \underbrace{\int P^M[g^{n-1}]\,dv}_{\leq 1}\, d\tau$$

$$\Downarrow$$
$$\left|\int Q^M(g^{n-1})\,dv\right| = 0$$
$$+ \mu\Big(\underbrace{\int g^{n-1}(v)\,dv}_{\leq 1}\Big)^2$$

$\Longrightarrow$ if $\int g^{n-1}\, dv \leq 1$

then $\int g^n\, dv \leq 1$.

$\{g_n\}$ increasing monotonely + $\int g^n \leq 1$.

$\xrightarrow[\text{theorem}]{\text{Levi's}}$ $g = \lim_{n\to\infty} g^n$ exist and nonenegative, satisfy $\int g\, dv$

≤ 1

$\lim_{n\to\infty}\int g^n$

$\Longrightarrow$ g solves the ③ so that $\boxed{g = f^M}$ positive

In particular, $\int f^M dv \leq 1$, if $\int f_0(v)\,dv = 1$.

#

Theorem: There exists a unique positive solution $f^M \in C^1([0,T]; L^1)$ to initial value problem ② for arbitrary time $T \geq 0$, provided that $f_0 \geq 0$ and $\int f_0\,dv = 1$.

Furthermore, suppose in addition that $E(f_0) = \frac{1}{2}\int_{\mathbb{R}^3} |v|^2 f_0(v)\,dv$ and $H(f_0) = \int_{\mathbb{R}^3} f_0(v) \ln f_0(v)\,dv$ are initially finite. Then

$$\begin{cases} E(f_0) = E(f^M(t)) \\ H(f^M(t)) \leq H(f_0) \end{cases}$$