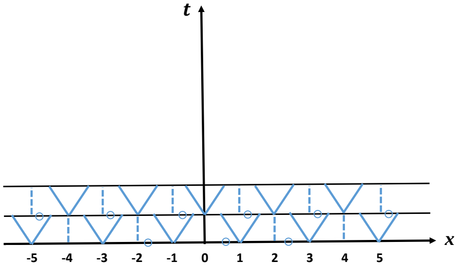
Section 5. Uniqueness of Glimm's Solution

§5.1 Notations for Glimm's Solution

- Choose mesh size $s = \Delta t$, $\gamma = \Delta x$, so that they satisfy CFL condition, $\dfrac{\gamma}{s} \geq \bar{\lambda}$.
- Let $\{\theta_l\}_{l=0}^{\infty}$ be a random sequence of numbers uniformly distributed on [-1,1].
- Construction of Glimm's scheme (by induction).

- Initially $0 < t < \Delta t = s$, we let $\nu(x,t)$ be the solution to

$$\begin{cases} \partial_t \nu + \partial_x f(\nu) = 0 \\ \nu(x, t=0) = \bar{u}((m+\theta)\Delta x),\ (m-1)\Delta x < x < (m+1)\Delta x, \\ \quad 0 < t < \Delta t,\ m = 0, \pm 2, \pm 4, \cdots \end{cases}$$



- Assume $\nu(x,t)$ has been constructed up to $t < l\Delta t$, we need to construct $\nu(x,t)$ on $l\Delta t < t < (l+1)\Delta t$ with initial data on $t = l\Delta t$ as

$$\nu(x, l\Delta t) = \nu((m+\theta_l)\Delta x, l\Delta t),$$

$$(m-1)\Delta x < x < (m+1)\Delta x, \quad m+l \quad \text{even}.$$

We repeat this construction with a given sequence $\{\theta_l\}_{l=0}^{\infty}$, but let mesh size $\gamma \to 0$ ($s \to 0$),

$$\{u_\nu\}_{\nu=1}^{\infty}, \qquad \nu \to +\infty \quad \text{corresponding to } \gamma \to 0.$$

By Glimm's functional, if $T.V.\,\bar{u} \ll 1$, then

$$T.V.\{u_\nu\} \le c_0\, T.V.\bar{u}$$

$\exists\{\nu_j\}$, such that $u_{\nu_j} \to u$ in L'_{loc}.

In general, u is not a weak solution (2.7), however, if $\{\theta_l\}_{l=0}^{\infty}$ is uniformly distributed on [-1,1], then u is an entropy weak solution to the Cauchy problem (2.7), (2.8).

Goal: The whole sequence u_ν coverge to u, and u is the unique limit, which depends continuously on initial data

$$u(\cdot, t) = S_t\, \bar{u}.$$

Theorem 5.1 Assume that

(1) The system (2.7) admits a SRSG.

(2) $T.V.\, u_0 \ll 1.$

(3) Let $\{u_\nu\}_{\nu=1}^{\infty}$ be a sequence of approximate solutions to (2.7) constructed by Glimm's method associated with a uniformly equally distributed sequence $\{\theta_l\}_{l=1}^{\infty} \subset [-1, 1]$ and $u = u(x,t) \in C([0,\infty);\ L'_{loc}(\mathbb{R}))$ is the a.e. limit of $\{u_\nu\}$, then $u = u(x,t) = S_t\, u_0$.

Proof of Theorem 5.1

Step 1: Some notations and facts:

- Let u be the limit of u_ν, $u \in C([0, \infty),\ L'(\mathbb{R}'))$

$$T.V.\, u(\cdot, t) \leq c\, T.V.\, u_0.$$

- $V_\nu(t)$: total variation of $u_\nu(t) \Leftrightarrow$ total amount of waves in $u_\nu(t)$.
- $Q_\nu(t)$: total amount of potential wave interaction of u_ν at time t.
- As proved before, $Q_\nu(t)$ is uniformly bounded and nonincreasing. So by monotone convergence theorem, there exists a subsequence $Q_\nu(t)$,

$$Q_\nu(t) \to Q(t), \qquad \forall\, t \geq 0.$$

- Clearly $Q(t)$ is bounded and nonincreasing, so $Q(t)$ maybe discontinuous at most at countably many times,

$$\mathcal{N} = \{\tau_1, \tau_2, \cdots ;\ \text{so that } Q(t) \text{ is discontinuous at } t_i\}.$$

  Our goal is to show that for any $\tau \in (0, \infty) \setminus \mathcal{N}$, that the inequalities in the definition of viscosity solution are satisfied with a suitably defined Radon measure μ_τ.
- For each i, $i = 1, 2, \cdots, n$ and $\nu \geq 1$. Let $\mu^\nu_{i\pm}$ be the unique Radon measures such that

$$\begin{aligned} \mu^\nu_{i+}(I) &= \text{the total amount of positive waves being in } I \text{ in } u_\nu(\cdot, \tau), \\ \mu^\nu_{i-}(I) &= \text{the total amount of negative waves lies in } I \text{ in } u_\nu(\cdot, \tau), \end{aligned}$$

  for any open interval I,

$$\mu^\nu_i = \mu^\nu_{i+} + \mu^\nu_{i-}.$$

Up to a subsequence if necessary, we can assume that there exist Radon measure $\mu_{i\pm}$ such that

$$\mu_{i\pm}^{\nu} \rightharpoonup \mu_{i\pm} \qquad \text{in measure.}$$

Set $\mu_i = \mu_{i+} + \mu_{i-},\ \mu_i^{\nu} \to \mu_i,$

$$\mu_\tau = \sum_{i=1}^{n} \mu_i,$$

μ_τ is a Radon measure.

Claim: The hypothesis in Proposition 4.2 is satisfied, i.e. there exists a uniform constant c and that $\forall$ fixed $(\xi, \tau) \in \mathbb{R}' \times ([0, \infty) \setminus \mathcal{N})$

$$
\text{(a)}\quad \frac{1}{\varepsilon}\int_{\xi-\rho+\bar\lambda\,\varepsilon}^{\xi+\rho-\bar\lambda\,\varepsilon} \left| u(x,\tau+\varepsilon) - U^{\#}_{u;\xi,\tau}(x,\tau+\varepsilon)\right| dx
\\ \leq c\,\mu_\tau((\xi-\rho,\xi)\cup(\xi,\xi+\rho))
$$

$$
\text{(b)}\quad \frac{1}{\varepsilon}\int_{\xi-\rho+\bar\lambda\,\varepsilon}^{\xi+\rho-\bar\lambda\,\varepsilon} \left| u(x,\tau+\varepsilon) - U^{b}_{u;\xi,\tau}(x,\tau+\varepsilon)\right| dx
\\ \leq c(\mu_\tau(\xi-\rho,\xi+\rho))^2
$$

as long as ε and ρ are sufficiently small.

Step 2: Local structure of Glimm's solutions (Glimm-Lax, Diferna).

Basic tools are:

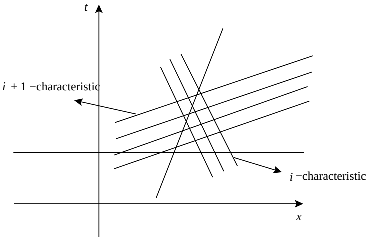
- Generalized characteristics
- Approximate conservation laws

- Let a segment γ in xt-plane be given as

$$\gamma(t) = \bar{x} + \lambda(t - \tau), \;\; t \in [\tau, \tau']$$

γ is non-characteristic if $\exists\, i$ such that

$$\lambda_i\left(u(x,t)\right) < \lambda < \lambda_{i+1}\left(u(x,t)\right), \quad \forall\,(x,t)$$



- Generalized characteristic curves

  Consider the following augmented system

$$\begin{cases} \partial_t u_0 + \lambda \partial_x u_0 = 0 \\ \partial_t u + \partial_x f(u) = 0 \end{cases}$$

$$U = (u_0, u) = (u_0, u_1, \cdots, u_n).$$

  This is a strictly hyperbolic system, since

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_i(u) < \lambda < \lambda_{i+1}(u) < \cdots < \lambda_n(u).$$

Definition 5.1 For a given solution $U = (u_0, u_1, u_n)$, then the j-th generalized characteristic curve is a Lipschitz continuous curve $t \mapsto \eta_j(t)$ such that

$$\dot{\eta}_j(t) = \lambda_j(U(\eta_j(t)+, t), U(\eta_j(t)-, t)).$$

- Approximate Conservation Laws
  Let $\Lambda \subset \mathbb{R}^2$ be a region bounded by either a space-like curve or a generalized characteristic curve. Let $\nu \geq 1$ be given, then one can also define the corresponding generalized characteristic curves associated with u_ν (approximate characteristic curve). Set

$$\begin{array}{rcl} E_j(\Lambda) & \overset{\Delta}{=} & E_j^+(\Lambda) + E_j^-(\Lambda), \text{ amount of } j\text{−waves entering } \Lambda. \\ L_j(\Lambda) & \overset{\Delta}{=} & L_j^+(\Lambda) + L_j^-(\Lambda), \text{ amount of } j\text{−waves leaving } \Lambda. \\ Q^+(\Lambda) & : & \text{total amount of wave interactions occurred in } \Lambda. \end{array}$$

Then the following estimates hold

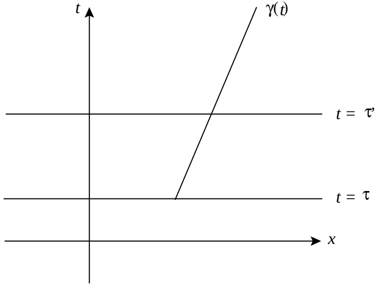
$$L_j(\Lambda) \leq E_j(\Lambda) + O(1)\, Q^+(\Lambda) \qquad \text{for} \quad j = 1, 2, \cdots, n.$$

Remark: The inequality is due to the possible cancellation of waves of the same family in Λ.

By taking limit, the approximate conservation laws hold for u.

Step 3: Some basic facts

- The estimate of j-waves crossing the non-characteristic segment.



For any closed interval I, we denote by

$$\begin{aligned} Q_\nu(\tau, I): &\quad \textit{potential of future wave interactions for } u_\nu \\ &\quad \textit{passing through } I \textit{ at } t=\tau. \\ Q_\nu(\tau, I) &\quad \longrightarrow Q(\tau, I) \end{aligned}$$

Lemma 5.1

(1) Let γ be the non-characteristic segment defined before.

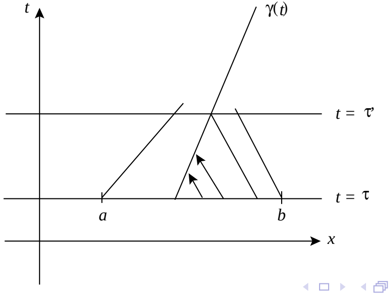
(2) Assume that all the generalized j-characteristic which cross γ originated at $t = \tau$ from some point in an interval $I \subseteq [a, b]$ and

$$\gamma(t) \in [a + \bar{\lambda}(t - \tau),\ b - \bar{\lambda}(t - \tau)], \qquad \forall\, t \in [\tau, \tau'].$$

Then

$$\begin{aligned} X_j(\gamma) &= O(1)\{\mu_j(I) + Q(\tau, [a, b])\} \\ &= O(1)\,\mu([a, b]) \end{aligned}$$

where $X_j(\gamma)$ is the total amount of j-waves crossing γ.



Proof of Lemma 5.1 For definiteness, we assume that $j \leq i$, thus all the generalized j-characteristics crossing γ from right to left. Starting from $(\gamma(\tau'), \tau')$, we draw the maximal backward generalized j-characteristic, $\eta_j(t)$. Define

$$\Lambda = \{(x,t) |\ \gamma(t) \leq x \leq \eta_j(t), \quad \tau \leq t \leq \tau'\},$$

we can apply the approximate conservation law to Λ,

$$\begin{aligned} X_j(\gamma) &\leq O(1)\,\mu_j(I) + O(1)\,Q^+(\Lambda) \\ &\leq O(1)\,\{\mu_j(I) + Q(\tau; [a,b])\} \end{aligned}$$

As a consequence of Lemma 5.1, one has the following lemma.

Lemma 5.2 Let the i-th family be linearly degenerate, i.e.

$$\nabla \lambda_i(u) \cdot \gamma_i(u) \equiv 0.$$

Then through each point (x_0, τ_0), there passes a unique i-th generalized characteristics which depends on (x_0, τ_0) continuously.

Remark: This lemma is the consequence of the a priori bound on the amount of wave crossing given in Lemma 5.1 and the assumption that the i-th family is linearly degenerate. The proof of the Lemma 5.2 is due to Bressan, based on the Lemma 5.1.

$\S$5.2 Wave structure and the wave-interaction potential

Goal: At each $t = \tau$, such that $Q(t)$ is continuous at $t = \tau$, then all the solutions to Riemann problem with $U(\xi\pm, \tau)$ are either a shock or contact discontinuity for all $\xi \in \mathbb{R}'$.

$(\mathcal{N} = \{t \in (0, \infty) \text{ such that } Q(t) \text{ is discontinuous}\} \text{ is countable})$

$$\tau \in (0, \infty) \backslash \mathcal{N}.$$

If this goal is achieved, then the verification of viscosity solution will be relatively easy. Indeed, we have

Lemma 5.3 Assume that $Q(t)$ is continuous at $t = \tau > 0$. Then

(1) $\forall\, \xi \in \mathbb{R}'$, $\exists\, i \in \{1, 2, \cdots, n\}$ such that

$$\mu_j(\{\xi\}) = 0, \qquad \forall\, j \neq i.$$

(2) If $\mu_i(\{\xi\}) > 0$ and i-th family is genuinely nonlinear, $\nabla \lambda_i \cdot \gamma_i > 0$, then for some $\tau' > \tau$, each approximate solution u^ν has an i-shock along an approximate characteristics $x = \eta_i^\nu(t)$, $t \in [\tau, \tau']$ with $\eta_i^\nu(\tau) \to \xi$ as $\nu \to +\infty$. In this case $\mu_{i-}(\{\xi\})$ is precisely the limit of the strength of the shock as $\nu \to \infty$, and $\mu_{i+}(\{\xi\}) = 0$.

Proof of Lemma 5.3 The key idea is to see the effects of wave interactions.

Step 1: (Proof of (1)) We would like to show that
$\exists\, i \in \{1, 2, \cdots, n\}$ such that $\mu_j(\{\xi\}) = 0$ if $j \neq i$.
If not, $\exists\, i$ and j such that $\mu_i(\{\xi\}) > 0$, $\mu_j(\{\xi\}) > 0$, $i \neq j$,
$\exists\, \delta > 0$, such that

$$\mu_i(\{\xi\}) > \delta > 0, \quad \mu_j(\{\xi\}) > \delta > 0$$

$\exists\, \delta_0 > 0$, and N such that

$$\mu_i^\nu(\{\xi\}) > \delta_0, \qquad \mu_j^\nu(\{\xi\}) > \delta_0, \qquad \forall\, \nu \geq N.$$

$\forall\, \varepsilon > 0$, in the interval $(\xi - \varepsilon, \xi + \varepsilon)$, the amount of i-waves and j-waves in $(\xi - \varepsilon, \xi + \varepsilon)$ will be bigger than δ_0, $\nu \geq N$.

Due to strict hyperbolicity, $|\lambda_i - \lambda_j| \geq m_0 > 0$, it is clear that these waves have to interact in the interval $[\tau - c\,\varepsilon, \tau + c\,\varepsilon]$, c depends only on ε.

$$\lim_{\varepsilon \to 0} |[Q(\tau + c\,\varepsilon) - Q(\tau - c\,\varepsilon)]| \geq \frac{\delta_0^2}{4}.$$

since ε is arbitrary, this contradicts with the fact that $Q(t)$ is continuous at $t = \tau$.

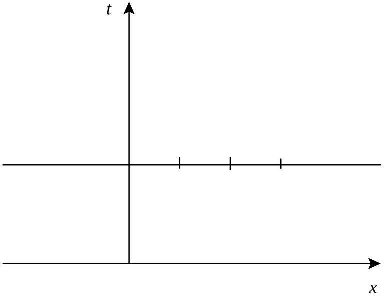
Step 2: Assume that $i \in \{1, 2, \cdots, n\}$ such that

$$\mu_i(\{\xi\}) > 0 \qquad \text{and} \qquad \nabla \lambda_i \cdot \gamma_i > 0.$$

We would like to show (2). If not, then $\exists\, \delta > 0$, such that for every $\varepsilon > 0$, $\exists\, \nu(\varepsilon)$ large enough, such that one of the followings occurs.

Case 1: The amount of i-shock and i-rarefaction waves in $u^\nu(\tau)$ contained in the interval $[\xi - \varepsilon, \xi + \varepsilon]$ are both $> \delta$.

Case 2: One can partition $[\xi - \varepsilon, \xi + \varepsilon]$ as $J_1^\nu \cup J_2^\nu$ such that $J_1^\nu \cap J_2^\nu = \phi$ and each J_k^ν contains amount of i-shock $> \delta$.

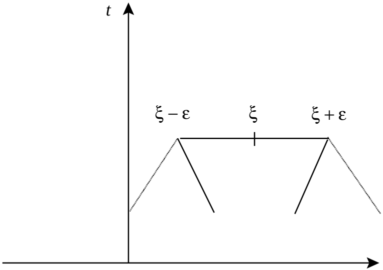


In Case 1 and Case 2, as $\nu \to +\infty$, an uniformly positive amount of interactions would take place in the u^ν, within a time of interval $[\tau, \tau + c\,\varepsilon]$, c is uniform independent of ε. (However, c may depends on δ, this is due to entropy condition.) Thus

$$\begin{aligned} \lim_{\varepsilon\to 0} \lim_{\nu\to\infty} |Q^\nu(\tau) - Q^\nu(\tau + c\,\varepsilon)| &\geq \frac{\delta^2}{10} \\ \overline{\lim}\,[Q(\tau) - Q(\tau+)] &\geq \frac{\delta^2}{10} \qquad \text{contradiction.} \end{aligned}$$

<u>Case 3:</u> $\mu_{i+}(\{\xi\}) > 0$. We show that this is impossible. Indeed, $\forall\,\varepsilon > 0$. Let $\eta_i^-(t)$ minimal backward i-th characteristic through $(\xi - \varepsilon, \tau)$.

$\eta_i^+(t)$ maximal backward i-th characteristics through $(\xi + \varepsilon, \tau)$.



Fact: $\dot{\eta}_i^+ - \dot{\eta}_i^-$ is proportional to the total amount of i-rarefaction waves in the interval $[\eta_i^-, \eta_i^+] \geq \delta > 0$ if no interactions take place. Then $\eta_i^\pm(t)$ will meet in the interval $[\tau - c\,\varepsilon, \tau]$, c might depend on δ but independent of ε. This is impossible.

Thus interactions must take place with $[\tau - c\,\varepsilon, \tau]$, then this will contradicts the continuity of $Q(t)$ at $t = \tau$.

Corollary 5.1 $Q(\tau, I) = O(1)\,\mu(I \setminus \{\xi\})$

Step 3: Verification of the conditions for Viscosity Solutions

Lemma 5.4 Let μ be a finite positive Radon measure defined on the interval $[a, b]$, assuming that $\lambda > 0$, $\delta > 0$, $\lambda' \in \mathbb{R}^1$. For $t > 0$, we define

$$\begin{aligned} \varphi(x) &= \mu((x - \lambda\, t,\ x + \lambda\, t]) \\ \psi(x) &= \mu((x - (\lambda' + \delta)t,\ x - (\lambda' - \delta)t]) \end{aligned}$$

Assume that $[\lambda' - \delta,\ \lambda' + \delta] \subset [-\lambda, \lambda]$. Then the following estimates hold:

$$
\begin{array}{llcl}
(1) & \displaystyle\int_{a+\lambda t}^{b-\lambda t} \varphi(x)\,dx & \le & 2\,\lambda\,t\,\mu((a,b)) \\
(2) & \displaystyle\int_{a+\lambda t}^{b-\lambda t} \varphi^2(x)\,dx & \le & 2\,\lambda\,t(\mu((a,b)))^2 \\
(3) & \displaystyle\int_{a+\lambda t}^{b-\lambda t} \psi(x)\,dx & \le & 2\,\delta\,t\,\mu((a,b))
\end{array}
$$

Proof $U(x) = \mu((a,x))$

For almost all x, $U(x+\lambda\,t) - U(x-\lambda\,t) = \varphi(x)$ and $U(x-(\lambda'-\delta)t) - U(x-(\lambda'+\delta)t) = \psi(x)$. It is then trivial to show (1), (2) and (3) by direct calculations.

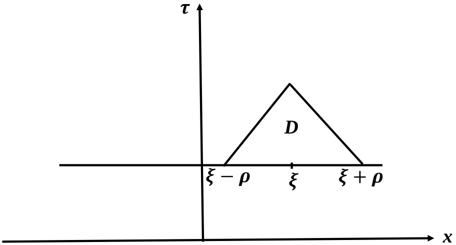
Step 3.1

$$\frac{1}{\varepsilon}\int_{\xi-\rho+\bar\lambda\,\varepsilon}^{\xi+\rho-\bar\lambda\,\varepsilon} \left|u(x,\tau+\varepsilon)-U^b_{(u;\xi,\tau)}(x,\tau+\varepsilon)\right|dx \le \mu_\tau((\xi-\rho,\xi+\rho))^2$$

for all $\xi\in\mathbb{R},\ \ \tau\in(0,\infty)\setminus\mathcal{N},\ \ 0<\rho,\ \ \varepsilon\ll 1.$

Now, for fixed $\xi,\ \ \rho>0,\ \ I_{\xi,\rho}=(\xi-\rho,\xi+\rho)$,

$$\begin{array}{rcl} D &=& \left\{(x,t)\ \middle|\ t\in\left[\tau,\tau+\frac{\rho}{\bar\lambda}\right],\ x\in(\xi-\rho+\bar\lambda(t-\tau),\ \xi+\rho-\bar\lambda(t-\tau))\right\} \\ \tilde u &=& u(\xi,\tau)\ =\ u(\xi+,\tau) \\ \tilde A &=& Df(\tilde u),\ \tilde\lambda_i,\ \tilde l_i,\ \tilde\gamma_i \end{array}$$



Let (u^-, u^+) be a single wave of the i-th family with strength σ, then

$$
\begin{aligned}
< \tilde{l}_i,\ u^+ - u^- > &= O(1)\,\sigma \\
< \tilde{l}_j,\ u^+ - u^- > &= O(1)\,\sigma \cdot \max\{|u^+ - \tilde{u}|, |u^- - \tilde{u}|\}
\end{aligned}
$$

Let $x = \gamma(s)$, $\forall\, s \in [\tau, t]$ be a non-characteristic segment contained in D. Then

$$
\begin{aligned}
(1) \quad & < \tilde{l}_i, u(\gamma(t), t) - u(\gamma(\tau), \tau) > \\
& = O(1) \left\{ X_i(\gamma) + \sup_{(x,t) \in D} |u(x,t) - \tilde{u}| \sum_{j \neq i} X_j(\gamma) \right\} \\
(2) \quad & |u(x,t) - u(\xi, \tau)| = O(1)\, \mu(I_{\xi,\rho}), \forall (x,t) \in D
\end{aligned}
$$

If follows from (2) that $\exists\, \delta^* > 0$ constant such that

$$0 < \delta^* = O(1)\, \mu(I_{\xi,\rho})$$

$$\tilde{\lambda}_i - \delta^* < \lambda_i(u_\nu(x,t),\ u_\nu(x',t')) < \tilde{\lambda}_i + \delta^* < \min \lambda_{i+1}, \quad \forall\, (x,t),\ (x',t') \in D$$

Since

$$
\begin{aligned}
& |\lambda_i(u(x,t),\ u(x',t')) - \tilde{\lambda}_i| \\
= \ & |\lambda_i(u(x,t),\ u(x',t')) - \lambda_i(u(\xi,\tau),\ u(\xi,\tau))| \\
\leq \ & O(1)(|u(x,t) - u(\xi,\tau)| + |u(x',t') - u(\xi,\tau)| \\
= \ & O(1)\, \mu(I_{\xi,\rho})
\end{aligned}
$$

Consider two linear problems

$$
\begin{cases}
\partial_t\, v + \tilde{A}\ \partial_x\, v = 0 \\
v(x, t = \tau) = u(x,\tau)
\end{cases}
\qquad t > \tau
$$

$$
\begin{cases}
\partial_t\, w + (\tilde{A} + \delta^* I)\ \partial_x\, w = 0 \\
w(x, t = \tau) = u(x,\tau)
\end{cases}
\qquad t > \tau
$$

$$\nu(x,t) = \sum_i < \tilde{l}_i,\ u(x - \tilde{\lambda}_i(t-\tau), \tau) > \tilde{\gamma}_i$$

$$w(x,t) = \sum_i < \tilde{l}_i,\ u(x - (\tilde{\lambda}_i + \delta^*)(t-\tau), \tau) > \tilde{\gamma}_i$$

so,

$$\begin{aligned}
&\int_{\xi-\rho+\bar{\lambda}(t-\tau)}^{\xi+\rho-\bar{\lambda}(t-\tau)} |w(x,t) - \nu(x,t)| dx \\
\leq & \sum_i \int_{\xi-\rho+\bar{\lambda}(t-\tau)}^{\xi+\rho-\bar{\lambda}(t-\tau)} |< \tilde{l}_i,\ u(x - \tilde{\lambda}_i(t-\tau), \tau) - u(x - (\tilde{\lambda}_i + \delta^*)(t-\tau), \tau) > \tilde{\gamma}_i| \\
\leq & \quad O(1) \sum_i \int_{\xi-\rho+\bar{\lambda}(t-\tau)}^{\xi+\rho-\bar{\lambda}(t-\tau)} \mu_\tau((x - (\tilde{\lambda}_i + \delta^*)(t-\tau),\ x - \tilde{\lambda}_i(t-\tau)]) \\
\leq & \quad O(1)\, \delta^*(t-\tau)\ \mu_\tau(I_{\xi,\rho}) \\
= & \quad O(1)\, (t-\tau)\ (\mu_\tau(I_{\xi,\rho}))^2
\end{aligned}$$

so it will be sufficient to show that

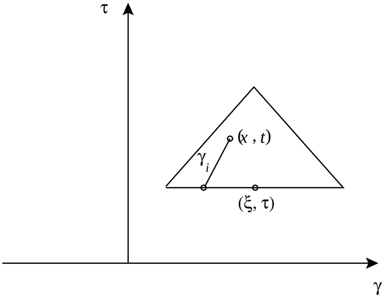
$$\int_{\xi-\rho+\bar{\lambda}(t-\tau)}^{\xi+\rho-\bar{\lambda}(t-\tau)} |u(x,t)-w(x,t)|dx \leq O(1)\,(t-\tau)\,\left(\mu(I_{\xi,\rho})\right)^2$$

For each fixed (x,t), we consider a non-characteristic segment

$$\gamma_i(s) = \gamma_i^{(x,t)}(s) = x - (\tilde{\lambda}_i + \delta^*)(t-s), \qquad \tau \leq s \leq t.$$

This is non-characteristic due to the choice of δ^*. Then

$$< \tilde{l}_i,\ u(x,t) - w(x,t) > = < \tilde{l}_i,\ u(x,t) - u(\gamma_i(\tau),\tau) > .$$



Applying (1), we have

$$
\begin{aligned}
& < \tilde{l}_i,\ u(x,t) - w(x,t) > = < \tilde{l}_i, u(x,t) - u(\gamma_i^{(x,t)}(\tau),\tau) > \\
= \ & O(1)\ \left\{ X_i(\gamma_i) + \sup_{(x,t)\in D} |u(x,t) - \tilde{u}| \sum_{j\neq i} X_j(\gamma_i) \right\} \\
= \ & O(1)\{\mu([x - (\tilde{\lambda}_i + \delta^*)(t-\tau),\ x - (\tilde{\lambda}_i - \delta^*)(t-\tau)]) \\
& + Q(t[x - \bar{\lambda}(t-\tau),\ x + \bar{\lambda}(t-\tau)])\} \\
& + O(1)\ \mu(I_{\xi,\rho})\ \mu([x - \bar{\lambda}(t-\tau),\ x + \bar{\lambda}(t-\tau)])
\end{aligned}
$$

By definition: $Q(\tau, I) \leq (\mu(I))^2$, so above estimates and Lemma 5.4 imply

$$
\begin{aligned}
& \int_{\xi-\rho+\bar\lambda(t-\tau)}^{\xi+\rho-\bar\lambda(t-\tau)} |u(x,t) - w(x,t)| dx \\
\leq\ & O(1) \sum_i \int_{\xi-\rho+\bar\lambda(t-\tau)}^{\xi+\rho-\bar\lambda(t-\tau)} |< \tilde{l}_i,\ u(x,t) - w(x,t) > | dx \\
\leq\ & O(1) \sum_i \left\{ \int_{\xi-\rho+\bar\lambda(t-\tau)}^{\xi+\rho-\bar\lambda(t-\tau)} \psi(x)\, dx + \int_{\xi-\rho+\bar\lambda(t-\tau)}^{\xi+\rho-\bar\lambda(t-\tau)} \varphi^2(x)\, dx \right. \\
& \left. + O(1)\, \mu(I_{\xi,\rho}) \int_{\xi-\rho+\bar\lambda(t-\tau)}^{\xi+\rho-\bar\lambda(t-\tau)} \varphi(x)\, dx \right\} \\
\leq\ & O(1) \big\{ 2\delta^*(t-\tau)\, \mu_\tau((\xi-\rho, \xi+\rho)) + 2\bar\lambda(t-\tau)(\mu_\tau((\xi-\rho,\xi+\rho)))^2 \\
& + O(1)\, \mu_\tau(I_{\xi,\rho})\ 2\bar\lambda(t-\tau)\ \mu_\tau(I_{\xi,\rho}) \big\} \\
\leq\ & O(1)\, (t-\tau)\, (\mu_\tau(I_{\xi,\rho}))^2
\end{aligned}
$$

Step 3.2: $\frac{1}{\varepsilon}\int_{\xi-\rho+\bar{\lambda}\varepsilon}^{\xi+\rho-\bar{\lambda}\varepsilon} |u(x,\tau+\varepsilon) - U^{\#}_{(u;\xi,\tau)}(x,\tau+\varepsilon)|dx \leq$ $c\,\mu_\tau((\xi-\rho,\xi)\cup(\xi,\xi+\rho))$

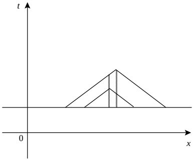
Case 1: $\mu(\{\xi\}) = \mu_\tau(\{\xi\}) = 0$. Thus $u(x,\tau)$ is continuous at $x=\xi$. Then

$$\mu(I_{\xi,\rho}) = \mu(I_{\xi,\rho}\setminus\{\xi\}) = \mu((\xi-\rho,\xi)\cup(\xi,\xi+\rho))$$

$$|u(x,t)-u(x,\tau)| = O(1)\,\mu([x-\bar{\lambda}(t-\tau),\ x+\bar{\lambda}(t-\tau)]) \qquad \forall\,(x,t)\in D$$

So it follows from the definition of $U^{\#}$, that

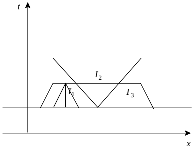
$$
\begin{aligned}
&\int_{\xi-\rho+\bar{\lambda}(t-\tau)}^{\xi+\rho-\bar{\lambda}(t-\tau)} |u(x,t) - U^{\#}_{u,\xi,\tau}(x,t)|dx \\
= \quad & I_1 + I_2 + I_3
\end{aligned}
$$



with

$$I_1 = \int_{\xi-\rho+\bar{\lambda}(t-\tau)}^{\xi-\bar{\lambda}(t-\tau)} |u(x,t) - u(x,\tau)|dx,$$

$$I_2 = \int_{\xi-\bar{\lambda}(t-\tau)}^{\xi+\bar{\lambda}(t-\tau)} |u(x,t) - \tilde{u}| dx,$$



$$I_3 = \int_{\xi+\bar{\lambda}(t-\tau)}^{\xi+\rho-\bar{\lambda}(t-\tau)} |u(x,t) - u(x,\tau)| dx,$$

Choose $\tilde{x}$ such that the straight line between $(\tilde{x}, t)$ and (ξ, τ) is non-characteristic, then

$$\begin{aligned} I_2 &= \int_{\xi-\bar{\lambda}(t-\tau)}^{\xi+\bar{\lambda}(t-\tau)} |u(x,t) - \tilde{u}|dx \\ &\leq \int_{\xi-\bar{\lambda}(t-\tau)}^{\xi+\bar{\lambda}(t-\tau)} (|u(x,t) - u(\tilde{x},t)| + |u(\tilde{x},t) - u(\xi,\tau)|)dx \\ &\leq O(1)\, \mu(I_{\xi,\rho})\ 2\bar{\lambda}(t-\tau) \end{aligned}$$

$$\begin{aligned} I_1 &= \int_{\xi-\rho+\bar{\lambda}(t-\tau)}^{\xi-\bar{\lambda}(t-\tau)} |u(x,t) - u(x,\tau)|dx \\ &\leq O(1) \int_{\xi-\rho+\bar{\lambda}(t-\tau)}^{\xi-\bar{\lambda}(t-\tau)} \mu_\tau((x-\bar{\lambda}(t-\tau),\ x+\bar{\lambda}(t-\tau)))dx \\ &= O(1)\, 2\bar{\lambda}(t-\tau)\ \mu_\tau((\xi-\rho,\xi)) \end{aligned}$$

Similarly,

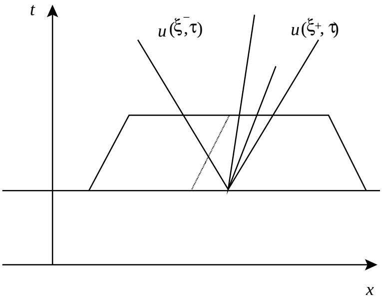
$$I_3 \le O(1)\, 2\bar{\lambda}(t-\tau)\ \mu_\tau((\xi, \xi+\rho)).$$

Thus,

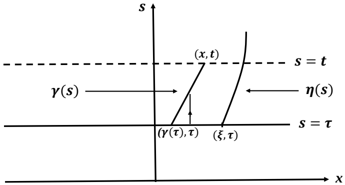
$$I_1 + I_2 + I_3 \le O(1)(t-\tau)\mu_\tau((\xi-\rho, \xi) \cup (\xi, \xi+\rho))$$

<u>Case 2:</u> $\mu(\{\xi\}) > 0$, then $\exists\, i$, such that $\mu(\{\xi\}) = \mu_i(\{\xi\}) > 0$, and the i-th family is genuinely nonlinear, i.e. $\nabla\lambda_i \cdot \gamma_i > 0$. Then in this case, the Riemann problem with data $(u(\xi^-, \tau),\ u(\xi+, \tau))$ is solved by an i-shock.

For the solution $u(x,t)$, there is a single i-th shock grows out at (ξ,τ), denoted by $x=\eta(t), t\in[\tau,\tau']$.



Let's assume that $(x, t) \in D$, $x < \eta(t)$. Let $s \to \gamma(s) = x - \bar{\lambda}(t - s)$, $t \in [\tau, t]$.



This is a non-characteristic segment. It should be clear that all i-waves which cross γ intersect with the line $t = \tau$ with a compact interval contained in $(\xi - \rho, \xi)$. (This is true due to entropy condition.)

Then

$$
\begin{aligned}
|u(x,t) - u(\xi-,\tau)| &\leq |u(x,t) - u(\gamma(\tau),\tau)| + |u(\gamma(\tau),\tau) - u(\xi-,\tau)| \\
&\leq O(1)\left\{ X_i(\gamma) + \sum_{j\neq i} X_j(\gamma) + \mu_\tau((\xi-\rho,\xi)) \right\} \\
&\leq O(1)\left\{ \mu_\tau((\xi-\rho,\xi)) + Q(\tau, I_{\xi,\rho}) + \sum_{j\neq i} \mu_j(I_{\xi,\rho}) \right\}
\end{aligned}
$$

$$
\begin{aligned}
\mu_j(I_{\xi,\rho}) &= \mu_j(I_{\xi,\rho}\setminus\{\xi\}) \\
Q(\tau,\ I_{\xi,\rho}) &= O(1)\,\mu_\tau(I_{\xi,\rho}\setminus\{\xi\})
\end{aligned}
$$

In conclusion, we have shown that

$$|u(x,t) - u(\xi-,\tau)| = O(1)\,\mu_\tau(I_{\xi,\rho} \backslash \{\xi\}) \quad \text{for} \quad (x,t) \in D,\ x < \eta(t).$$

Exactly, we have

$$|u(x,t) - u(\xi+,\tau)| = O(1)\,\mu_\tau(I_{\xi,\rho} \setminus \{\xi\}) \quad \forall (x,t) \in D, x > \eta(t).$$

As a consequence, we get

$$|\eta'(t) - \lambda_i(u(\xi-,\tau),\ u(\xi+,\tau))| = O(1)\,\mu_\tau(I_{\xi,\rho} \setminus \{\xi\})$$

$$\int_{\xi-\rho+\bar\lambda(t-\tau)}^{\xi+\rho-\bar\lambda(t-\tau)} |u(x,t) - U^{\#}_{u,\xi,\tau}(x,t)|dx = \int_{I_1 \cup I_2 \cup I_3} |u - U^{\#}|$$

I_1 is bounded by $\xi + \lambda_i(u(\xi-, \tau),\ u(\xi+, \tau))(t - \tau),\ \eta(t),$

$$I_2 = [\xi - \bar{\lambda}(t - \tau),\ \xi + \bar{\lambda}(t - \tau)] \setminus I_1 = I_2^+ \cup I_2^-,$$

with $I_2^- = I_2 \cap \{x < \eta(t)\}, I_2^+ = I_2 \cap \{x > \eta(t)\},$

$$I_3 = I \setminus \{I_1 \cup I_2\}.$$

$$\int_{I_1} |u(x,t) - U^{\#}_{u,\xi,\tau}(x,t)|dx = O(1)\ |I_1|$$
$$= O(1)\, \mu_\tau(I_{\xi,\rho} \setminus \{\xi\})(t-\tau)$$

$$\int_{I_2^-} |u(x,t) - U^{\#}_{u,\xi,\tau}(x,t)|dx = \int_{I_2^-} |u(x,t) - u(\xi-,\tau)|dx$$
$$= O(1)\, \mu(I_{\xi,\rho} \setminus \{\xi\})\ |I_2^-|$$
$$= O(1)\, \mu(I_{\xi,\rho} \setminus \{\xi\})\ 2\bar{\lambda}(t-\tau)$$

Similarly,

$$\int_{I_2^+} |u(x,t) - U^{\#}_{u,\xi,\tau}(x,t)|dx \leq O(1)\, \mu(I_{\xi,\rho} \setminus \{\xi\})\ 2\bar{\lambda}(t-\tau).$$

The integral over I_3 is the same as Case 1.

Case 3: $\mu(\{\xi\}) = \mu_i(\{\xi\}) > 0$ and the i-th family is linearly degenerate. Then the Riemann problem with the data $(u(\xi-,\tau),\ u(\xi+,\tau))$ is a contact discontinuity. And furthermore, in $u(x,t)$, there exists contact discontinuity $x = \eta(t)$ growing out of (ξ,τ), $\tau < t < \tau'$ for some $\tau' > \tau$. Note that $x = \eta(t)$ is the unique i-th generalized characteristic curve through (ξ,τ).

Claim: All the analysis in Case 2 is true for this case. (exercise)