Mid-term Practice Problems 1

Problem 1

Prove that if G has order 1365 or 6545, then G is not simple.

Problem 2

Let G be a finite group and p a prime dividing the order of G. Let K be the set of all elements of G whose order is a power of p. Prove that K is a subgroup if and only if there exists a unique Sylow p-subgroup.

Problem 3

Let G be a group and let N be a normal subgroup of index n. Show that $g^n \in N$ for all $g \in G$.

Problem 4

Show that if G is a non-abelian finite group, then $|Z(G)| \leq 1/4|G|$.

Problem 5

Prove that the commutator subgroup of $SL_2(\mathbb{Z})$ is proper in $SL_2(\mathbb{Z})$.

Problem 6

(a) Find the centralizer in S_7 of $(123)(4567)$.

(b) How many elements of order 12 are there in S_7?

Problem 7

Prove that the symmetric group S_n is a maximal subgroup of S_{n+1}.

Problem 8

Let G be a group of order 16 with an element g of order 4. Prove that the subgroup of G generated by g^2 is normal in G.

Problem 9

We say that a group X is involved in a group G if X is isomorphic to H/K for some subgroups K, H of G with $K \trianglelefteq H$. Prove that if X is solvable and X is involved in the finite group G, then X is involved in a solvable subgroup of G.

Problem 10

Let G be a finite group and let N be a normal subgroup of G with the property that G/N is nilpotent. Prove that there exists a nilpotent subgroup H of G satisfying $G = HN$.