# 1 Linear, Nonlinear, Linear Homogeneous, and Linear Inhomogeneous

**Exercise 1.1**

For each of the following equation, determine whether it is nonlinear or linear. If it is linear, determine whether it is homogeneous or inhomogeneous.

1. $u_x + e^y u_y = 0$;
2. $u_x + u_y + 1 = 0$;
3. $u_x + \left(\frac{u^2}{2}\right)_y = 0$;
4. $u_t - u_{xx} + u^3$=0.

**Solutions:**

1. linear homogeneous.

2. linear inhomogeneous.

3. nonlinear. Since $\mathscr{L}u = u_x + uu_y$, and

$$\begin{aligned}\mathscr{L}(u+v) &= (u+v)_x + (u+v)(u+v)_y \\ &= \mathscr{L}u + \mathscr{L}v + uv_y + vu_y \\ &\neq \mathscr{L}u + \mathscr{L}v,\end{aligned}$$

$\mathscr{L}$ is not a linear operator. Note that the nonlinear part is $uu_y$.

4. nonlinear. Since $\mathscr{L}u = u_t - u_{xx} + u^3$,

$$\begin{aligned}\mathscr{L}(u+v) &= (u+v)_t - (u+v)_{xx} + (u+v)^3 \\ &= \mathscr{L}u + \mathscr{L}v + 3u^2v + 3uv^2 \\ &\neq \mathscr{L}u + \mathscr{L}v,\end{aligned}$$

$\mathscr{L}$ is not a linear operator. Note that the nonlinear part is $u^3$.

**Remark 1.1**

In practice, there are more specific classifications of nonlinear PDEs. For equation 4, we could observe that if we regard the nonlinear part $u^3$ as a function of $x$ and $t$, then it is a linear inhomogeneous equation. This type of nonlinear equations is usually called semilinear PDEs, usually with the form $\mathscr{L}u = f(u)$ where $\mathscr{L}$ is a linear operator. For equation 3, if we regard the nonlinear part $uu_y$ as $f(x)u_y$, then it is a linear equation (transport equation). This type of nonlinear equations is usually called quasilinear PDEs, usually with the form $\mathscr{L}(u, \partial u, \ldots)u = 0$, where $\mathscr{L}(u, \partial u, \ldots)$ means that the coefficients of the linear operator depend on $u$, $\partial u$, $\ldots$, with lower orders than the terms in the linear operator.

## 2 Integration by Parts and Green's Formula

Firstly, we recall the divergence theorem:

$$\int_{\Omega} \operatorname{div} \vec{F} = \int_{\partial\Omega} \vec{F} \cdot \vec{n} \, \mathrm{d}S.$$

Suppose $\Omega \subset \mathbb{R}^n$ is bounded and open.

**Definition 2.1**

We say $\partial\Omega \in C^k$ if for each point $x \in \partial\Omega$, there is a $r > 0$ and a $C^k$ function $\gamma \colon \mathbb{R}^{n-1} \to \mathbb{R}$ such that, upon relabeling and reorienting the coordinate axes if necessary,

$$\Omega \cap B(x_0, r) = \{x \in B(x_0, r) \mid x_n > \gamma(x_1, \ldots, x_{n-1})\}.$$

$\partial\Omega \in C^\infty$ means that $\gamma$ is $C^\infty$.

This notion is related to the notion manifolds with boundary in differential geometry. Now we have a precise version of the divergence theorem.

**Theorem 2.1** *Suppose that $\partial\Omega \in C^1$ and $\vec{F} \in C^1(\overline{\Omega})$. Then*

$$\int_{\Omega} \operatorname{div} \vec{F} = \int_{\partial\Omega} \vec{F} \cdot \vec{n} \, \mathrm{d}S$$

*where $\vec{n}$ is the normal vector.*

If we let the $i$-th component of $F$ be $uv$ and other components be $0$, then we have the formula of integration by parts in high dimensions.

**Theorem 2.2** *Suppose that $\partial\Omega \in C^1$ and $u$, $v \in C^1(\overline{\Omega})$. Then*

$$\int_{\Omega} u_i v = -\int_{\Omega} u v_i + \int_{\partial\Omega} u v \vec{n}_i \, \mathrm{d}S.$$

This formula also have the function of transferring a derivative from one to another. Sometimes it is more flexible than the divergence theorem, because it just focus on one derivative.

As an application of integration by parts, next we introduce Green's formula as an exercise.

**Exercise 2.1**

Suppose that $\partial\Omega \in C^1$ and $u$, $v \in C^2(\overline{\Omega})$. Prove the following identities.

1. $\int_{\Omega} \Delta u = \int_{\partial\Omega} \frac{\partial u}{\partial \vec{n}} \, \mathrm{d}S$;

2. $\int_\Omega \nabla u \cdot \nabla v = -\int_\Omega u\Delta v + \int_{\partial\Omega} u\frac{\partial v}{\partial \vec{n}}\,\mathrm{d}S$;
3. $\int_\Omega (u\Delta v - v\Delta u) = \int_{\partial\Omega} \left(u\frac{\partial v}{\partial \vec{n}} - \frac{\partial u}{\partial \vec{n}}v\right)\mathrm{d}S$,

where $\frac{\partial f}{\partial \vec{n}} = \vec{n}\cdot\nabla f$ is the derivative of $f$ in the direction of the normal vector.

**Solutions:**

1. By integration by parts,
$$\int \Delta u = \int \sum_i u_{ii} = \int_{\partial\Omega} \sum_i u_i \vec{n}_i = \int_{\partial\Omega} \frac{\partial u}{\partial \vec{n}}.$$
2. By integration. by parts,
$$\int \nabla u\cdot\nabla v = \int \sum_i u_i v_i = -\int \sum_i u v_{ii} + \int_{\partial\Omega} \sum_i u v_i \vec{n}_i = -\int u\Delta v + \int_{\partial\Omega} u\frac{\partial v}{\partial \vec{n}}.$$
3. Note that the left hand side of 2 is symmetric with respect to $u$ and $v$. If we switch $u$ and $v$, we obtain that
$$\int \nabla u \cdot \nabla v = -\int v\Delta u + \int_{\partial\Omega} v\frac{\partial u}{\partial \vec{n}}.$$
By subtracting them, we could obtain 3. 3 could also be proved by integration by parts twice, which is a standard process. You could try it as an exercise to be familiar to integration by parts.

This section is mainly from Evans' PDE appendix C.1 and C.2. You may refer to it.