MATH4030 Tutorial Notes 4

Gauss Curvature and Mean Curvature

Classical minimal surfaces.

We say a surface $\boldsymbol{X}(u,v)$ is a minimal surface if the mean curvature is zero everywhere. We define the catenoid to be the following surface

$$\boldsymbol{X}(u,v) = (\cosh v \cos u, \cosh v \sin u, v), \quad u \in [0, 2\pi), v \in \mathbb{R}.$$

Compute the Gauss curvature, mean curvature, area of Gauss image of catenoid. Show that catenoid is a minimal surface.

Direct computation.

$$\begin{aligned}
\boldsymbol{X}_u &= (-\cosh v \sin u, \cosh v \cos u, 0)\\
\boldsymbol{X}_v &= (\sinh v \cos u, \sinh v \sin u, 1)\\
\boldsymbol{N} &= \frac{1}{\cosh v}(\cos u, \sin u, -\sinh v)\\
\boldsymbol{X}_{uu} &= -\cosh v(\cos u, \sin u, 0)\\
\boldsymbol{X}_{uv} &= \sinh v(-\sin u, \cos u, 0)\\
\boldsymbol{X}_{vv} &= \cosh v(\cos u, \sin v, 0)\\
\mathrm{I} &= \cosh^2 v du^2 + \cosh^2 v dv^2\\
\mathrm{II} &= -du^2 + dv^2\\
K &= \frac{eg-f^2}{EG-F^2} = -\frac{1}{\cosh^4 v}\\
H &= \frac{1}{2}\frac{eG-2fF+gE}{EG-F^2} = 0 \quad \text{(So it is a minimal surface)}\\
\text{Signed area}(\boldsymbol{N}) &= \int K dA = \int_0^{2\pi}\int_{-\infty}^{\infty} -\frac{1}{\cosh^4 v}\cosh^2 v dv du = -4\pi
\end{aligned}$$

Area of Gauss image of Tubular surfaces.

$\alpha(s) : I \to \mathbb{R}^3$ a curve parametrized by arc length.

$$\boldsymbol{X}(u,v) = \alpha(u) + rN(u)\cos v + rB(u)\sin v.$$

So

$$\begin{aligned}
\mathrm{I} &= \left((1 - r\kappa\cos v)^2 + r^2\tau^2\right) du^2 + 2r^2\tau du dv + r^2 dv^2\\
\mathrm{II} &= \left(\kappa\cos v(\kappa r\cos v - 1) + r\tau^2\right) du^2 + 2r\tau du dv + r dv^2\\
\int K dA &= \int_0^{2\pi}\int_I \frac{eg-f^2}{\sqrt{EG-F^2}} du dv = \int_0^{2\pi}\int_I \frac{r\kappa\cos v(\kappa r\cos v - 1)}{r(1 - r\kappa\cos v)} du dv\\
&= \int_0^{2\pi}\int_I -\kappa(u)\cos v du dv = 0
\end{aligned}$$

Minimal surface equations. Suppose $f(x,y):\Omega\to\mathbb{R}$ is a smooth function on a domain Ω. Then the mean curvature of graph f is given by

$$H=\frac{1}{2}\mathrm{div}\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}.$$

Moreover, the graph of f is a minimal surface if and only if

$$(1+f_y^2)f_{xx}-2f_xf_yf_{xy}+(1+f_x^2)f_{yy}=0.$$

Proof. We suppose

$$\boldsymbol{X}(u,v)=(u,v,f(u,v))$$

We already know

$$\begin{aligned}
\mathrm{I}&=(1+f_u^2)du^2+2f_uf_vdudv+(1+f_v^2)dv^2\\
\boldsymbol{N}&=\frac{1}{\sqrt{1+|\nabla f|^2}}(-f_u,-f_v,1)\\
\boldsymbol{X}_{ij}&=(0,0,f_{ij})\\
h_{ij}&=\frac{f_{ij}}{\sqrt{1+|\nabla f|^2}}\\
\mathrm{II}&=\frac{1}{\sqrt{1+|\nabla f|^2}}\left(f_{uu}du^2+2f_{uv}dudv+f_{vv}dv^2\right)\\
H&=\frac{1}{2\left(1+|\nabla f|^2\right)^{\frac{3}{2}}}\left(f_{uu}(1+f_v^2)+f_{vv}(1+f_u^2)-2f_uf_vf_{uv}\right)
\end{aligned}$$

On the other hand, we know

$$\begin{aligned}
\left(\frac{f_u}{\sqrt{1+|\nabla f|^2}}\right)_u&=\frac{f_{uu}}{\sqrt{1+|\nabla f|^2}}-\frac{f_u(f_uf_{uu}+f_vf_{vu})}{\left(1+|\nabla f|^2\right)^{\frac{3}{2}}}\\
&=\frac{f_{uu}(1+f_v^2)-f_uf_vf_{uv}}{\left(1+|\nabla f|^2\right)^{\frac{3}{2}}}
\end{aligned}$$

Similarly we can compute the derivative with respect to v. This will give

$$\mathrm{div}\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}=2H$$

□

Quick application

In particular, we have the following simple formula when $\nabla f=0$.

Proposition. *Suppose $\boldsymbol{X}(u,v)=(u,v,f(u,v))$ with $\nabla f(0,0)=0$. Then*

$$H_{(0,0)}=\frac{1}{2}\Delta f(0,0)$$

So locally, minimal surface looks like a harmonic function.

Mean curvature of spheres. Consider

$$f(u,v) = \sqrt{R^2 - u^2 - v^2}.$$

Then

$$f(u,v) = R\sqrt{1 - \frac{u^2+v^2}{R^2}} = R\left(1 - \frac{u^2+v^2}{2R^2} + o(u^2+v^2)\right)$$

So

$$H_{(0,0)} = \frac{1}{2}\Delta f(0,0) = -\frac{1}{R}$$