MATH4030 Tutorial Notes 3

Shape operator and second fundamental form

Tubular surfaces

Given a regular space curve $\alpha(s) : I \to \mathbb{R}^3$ with $\kappa(s) > 0$ parametrized by arc length, consider the tubular surface of α defined by

$$\boldsymbol{X}(u,v) = \alpha(u) + rN(u)\cos v + rB(u)\sin v, \quad r > 0 \text{ small enough}$$

Compute the second fundamental form of $\boldsymbol{X}$.

$$\begin{aligned}
\boldsymbol{X}_u &= (1-\kappa r\cos v)T + r\tau\cos vB - r\tau\sin vN \\
\boldsymbol{X}_v &= -r\sin vN + r\cos vB \\
\boldsymbol{N} &= -B\sin v - N\cos v \\
\boldsymbol{N}_u &= \tau N\sin v + \kappa T\cos v - \tau B\cos v \\
\boldsymbol{N}_v &= -B\cos v + N\sin v \\
e &= -\langle \boldsymbol{X}_u, \boldsymbol{N}_u\rangle = -\kappa\cos v\,(1-\kappa r\cos v) + r\tau^2 \\
f &= -\langle \boldsymbol{X}_u, \boldsymbol{N}_v\rangle = r\tau \\
g &= -\langle \boldsymbol{X}_v, \boldsymbol{N}_v\rangle = r
\end{aligned}$$

So the second fundamental form is

$$\mathrm{II} = \left(\kappa^2 r\cos^2 v - \kappa\cos v + r\tau^2\right)du^2 + 2r\tau dudv + rdv^2$$

Shape operator $\simeq$ the second derivative of surfaces

Consider the graph of function $f(u,v)$ defined by

$$\Sigma : \boldsymbol{X}(u,v) = (u, v, f(u,v))$$

Suppose $\nabla f(0,0) = 0$. Then $S_{(0,0)} : T_{(0,0)}\Sigma \to T_{(0,0)}\Sigma$ is indeed a linear transformation on the x-y plane. Compute the shape operator at $(0,0)$.

$$\begin{aligned}
\boldsymbol{X}_u &= (1, 0, f_u) \\
\boldsymbol{X}_v &= (0, 1, f_v) \\
\boldsymbol{N} &= \left(-\frac{f_u}{\sqrt{1+|\nabla f|^2}}, -\frac{f_v}{\sqrt{1+|\nabla f|^2}}, \frac{1}{\sqrt{1+|\nabla f|^2}}\right) \\
\boldsymbol{N}_u(0,0) &= (-f_{uu}, -f_{uv}, 0) \\
\boldsymbol{N}_v(0,0) &= (-f_{uv}, -f_{vv}, 0)
\end{aligned}$$

So

$$d\boldsymbol{N}_{(0,0)} : \boldsymbol{X}_u = (1,0,0) \to -(f_{uu}, f_{uv}, 0)\,, \quad \boldsymbol{X}_v = (0,1,0) \to -(f_{uv}, f_{vv}, 0)$$

Restrict on x-y plane, we have

$$d\boldsymbol{N}_{(0,0)} : \begin{bmatrix} x \\ y \end{bmatrix} \to -\text{Hess}(f) \begin{bmatrix} x \\ y \end{bmatrix}$$

So $S_{(0,0)} = \text{Hess}(f)$. Shape operator is exactly the second derivative of f. First and second fundamental form in this case.

$$\begin{aligned} \text{I}_{(0,0)} &= du^2 + dv^2 \\ \text{II}_{(0,0)} &= f_{uu}du^2 + 2f_{uv}dudv + f_{vv}dv^2. \end{aligned}$$

Examples mentioned in lecture class.
1. Upper half sphere. $f(x,y) = \sqrt{1-x^2-y^2}$.

$$\boldsymbol{X} = (u, v, \sqrt{1-u^2-v^2})$$

We have

$$\begin{aligned} \nabla f(x,y) &= \left(-\frac{x}{\sqrt{1-x^2-y^2}}, -\frac{y}{\sqrt{1-x^2-y^2}} \right) \\ \text{Hess}f(0,0) &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \end{aligned}$$

So $S_{(0,0)} = -\text{Id}$.
2. Hyperboloid. $f(x,y) = y^2 - x^2$.

$$\boldsymbol{X}(u,v) = (u, v, v^2 - u^2)$$

We have

$$S_{(0,0)} = \text{Hess}f(0,0) = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}.$$

Normal curvature of curves on surface and relation with the 2nd fundamental form.

Let $\alpha(s) = \boldsymbol{X}(u(s), v(s))$ be the regular curve on surface parametrized by arc length.

Recall that first fundamental form measures the length of vectors in tangent space, so

$$|\alpha'(s)| = 1 = \sqrt{Eu'^2 + 2Fu'v' + Gv'^2}.$$

The curvature vector of α is $\alpha''(s)$ and the curvature $\kappa = |\alpha''(s)|$. We define the normal curvature by following

$$k_n(s) := \langle \alpha''(s), \boldsymbol{N}(u(s), v(s)) \rangle$$

Proposition. *We can compute the normal curvature of a curve by second fundamental form*

$$k_n(s) = \text{II}(\alpha'(s), \alpha'(s)).$$

Moreover, the value of normal curvature of curvature does not rely the choice of curves, only depends on the tangent vectors.

Proof. Since $\langle \boldsymbol{N}(s), \alpha'(s) \rangle = 0$, taking derivative, we have

$$\langle \boldsymbol{N}(s), \alpha''(s) \rangle = - \left\langle \boldsymbol{N}'(s), \alpha'(s) \right\rangle .$$

So

$$\begin{aligned} k_n(s) = & \ \langle N(s), \alpha''(s) \rangle = - \left\langle \frac{d\boldsymbol{N}}{ds}, \alpha'(s) \right\rangle = - \langle d\boldsymbol{N}(\alpha'(s)), \alpha'(s) \rangle \\ = & \ \mathrm{II}(\alpha'(s), \alpha'(s)) \end{aligned}$$

For any two curves $\alpha(s), \beta(s)$ on surface with $\alpha'(0) = \beta'(0)$ (parametrized by arc length), we have

$$k_n^{\alpha}(0) = k_n^{\beta}(0)$$

which the normal curvature only depending on the tangent vector $\alpha'(0)$. $\square$

Example. Show that any regular curves in $\mathbb{S}^2$ has normal curvature -1.

Let $\alpha(s)$ be the regular curves. To compute the normal curvature, we need s is the arc length of α. So

$$k_n(s) = \mathrm{II}(\alpha', \alpha') = \langle -\mathrm{Id}(\alpha'), \alpha' \rangle = - \langle \alpha', \alpha' \rangle = -1.$$