MATH4030 Tutorial Notes 2

Regular surfaces

Theorem. *The level sets of regular values are regular surfaces.*

In general, the reverse might not be true.
Two examples.
1. Find the regular values of

$$F(x,y,z) = \left(x^2 + y^2 + z^2 - 1\right)^2$$

and the values of level set of F is a regular surface.
Solutions.

$$\nabla F(x,y,z) = 2(x^2 + y^2 + z^2 - 1)\,(2x, 2y, 2z)\,.$$

$F = C$ is well-defined only when $C \geq 0$. So $|\nabla F| \neq 0$ if and only if $x \neq 0, y \neq 0, z \neq 0$ and $x^2 + y^2 + z^2 \neq 1 \iff 0 < C < 1$ or $C > 1$. The regular values of F is $(0,1) \cup (1,\infty)$.

To find the values making the level sets of F to be a regular surface, we consider the following four cases.

- $C = 0$, $F = 0 \iff x^2 + y^2 + z^2 = 1$, level set is a unit sphere. It is a regular surface.
- $C \in (0,1) \iff x^2 + y^2 + z^2 = 1 + C$ or $x^2 + y^2 + z^2 = 1 - C$. Level set contains two spheres. It is a regular surface.
- $C = 1 \iff x^2 + y^2 + z^2 = 2$ or $x = y = z = 0$. Level set contains one sphere and a point. It is not a regular surface.
- $C > 1 \iff x^2 + y^2 + z^2 = 1 + C$. Level set is a sphere. It is a regular surface.

In summary, when $C \in [0,1) \cup (1,\infty)$, the level set of F is a regular surface.

2. Consider $F(x,y,z) = (z - x^2 - y^2)(x^2 + y^2 + z^2)$. Verify 0 is not a regular point only at point $(0,0,0)$. But $F = 0$ is still a regular surface.
Solutions.

$$\nabla F(x,y,z) = (x^2 + y^2 + z^2)(-2x, -2y, 1) + (z - x^2 - y^2)(2x, 2y, 2z)$$

When $F = 0$, we have $x = y = z = 0$ or $z = x^2 + y^2$. Actually, $(0,0,0)$ is contained in the graph of $z = x^2 + y^2$. So $F = 0 \iff z = x^2 + y^2$. This is a regular surface. In this case, we have

$$\nabla F(x,y,z) = (x^2 + y^2 + z^2)(-2x, -2y, 1)$$

when $z = x^2 + y^2$. So $|\nabla F| \neq 0$ if and only if $x \neq 0, y \neq 0, z \neq 0$.

Hence F is regular on $z = x^2 + y^2$ except $(0,0,0)$ but $F = 0$ is a regular surface even near $(0,0,0)$.

Non-regular surfaces

Consider the set defined by

$$S := \{z^2 = x^2 + y^2\}$$

Verify S is not a regular surface. But when we remove one point, $S \setminus \{(0,0,0)\}$ is indeed a regular surface.

Solutions. We only need to verify S is not a graph in any one of the coordinate plane locally.

Near $(0,0,0)$, we know S can be written as

- $z = \pm\sqrt{x^2 + y^2}$, not a graph on coordinate plane x-y.
- $y = \pm\sqrt{z^2 - x^2}$, not a graph on coordinate plane x-z.
- $x = \pm\sqrt{z^2 - y^2}$, not a graph on coordinate plane y-z.

So S is not a regular surface.

But when we remove one point, we can consider

$$F(x, y, z) := z^2 - x^2 - y^2$$

We know

$$\nabla F(x, y, z) = (2z, -2x, -2y)$$

So $|\nabla F| \neq 0$ on $S \setminus \{(0,0,0)\}$. Hence, $S \setminus \{(0,0,0)\}$ is a regular surface.

First fundamental form of tubular surfaces

Given a regular space curve $\alpha(s) : I \to \mathbb{R}^3$ with $\kappa(s) > 0$ parametrized by arc length, consider the tubular surface of α defined by

$$\boldsymbol{X}(u, v) = \alpha(u) + rN(u)\cos v + rB(u)\sin v, \quad r > 0 \text{ small enough}$$

Compute the normal of $\boldsymbol{X}$ and the first fundamental form of $\boldsymbol{X}$. Moreover, the area of this surface is just 2π times the length of α.

$$\begin{aligned}
\boldsymbol{X}_u &= T + r(-\kappa T\cos v + \tau B\cos v - \tau N\sin v)\\
\boldsymbol{X}_v &= r(-N\sin v + B\cos v)\\
E &= \langle \boldsymbol{X}_u, \boldsymbol{X}_u\rangle = (1 - r\kappa\cos v)^2 + r^2\tau^2\\
F &= \langle \boldsymbol{X}_u, \boldsymbol{X}_v\rangle = r^2\tau\\
G &= \langle \boldsymbol{X}_v, \boldsymbol{X}_v\rangle = r^2\\
\boldsymbol{N} &= \frac{\boldsymbol{X}_u \times \boldsymbol{X}_v}{|\boldsymbol{X}_u \times \boldsymbol{X}_v|} = \frac{(1 - r\kappa\cos v)rT \times (-N\sin v + B\cos v)}{|(1 - r\kappa\cos v)rT \times (-N\sin v + B\cos v)|} = -B\sin v - N\cos v
\end{aligned}$$

So the first fundamental form is

$$I = \left((1 - r\kappa\cos v)^2 + r^2\tau^2\right)du^2 + 2r^2\tau\, du dv + r^2 dv^2$$

Direct computation of area.

$$\begin{aligned}
\text{Area}(\boldsymbol{X}) &= \int_0^{2\pi}\int_I \sqrt{EG - F^2}\,dudv = \int_0^{2\pi}\int_I r(1 - r\kappa\cos v)\,dudv\\
&= 2\pi r\int_I 1du = 2\pi r\text{Length}(I) = 2\pi r\text{Length}(\alpha).
\end{aligned}$$