MATH4030 Tutorial Notes 1

Curvature for planar curves

Definition of signed curvature for planar curves

Let $\alpha(s) : I \subset \mathbb{R} \to \mathbb{R}^2$ be a curve parametrized by arc-length. $J : \mathbb{R}^2 \to \mathbb{R}^2$ be a counterclockwise rotation by $\frac{\pi}{2}$ on $\mathbb{R}^2$.

$T(s) := \alpha'(s)$, tangent vector.

$N(s) := J(T(s))$. (Not same with $\mathbb{R}^3$ curves.)

We call $\{T(s), N(s)\}$ the Frenet frame along α. $\{T(s), N(s)\}$ will form a orthonormal basis for $\mathbb{R}^2$ for any $s \in I$. It has the same orientation with the standard orthonormal basis $\{e_1, e_2\}$.

We define the signed curvature of $\alpha(s)$ to be

$$k(s) = \langle T'(s), N(s) \rangle .$$

It is well-defined even if $\alpha''(s) = 0$.

Differential of tangent and normal vector fields.

From $\langle T'(s), T(s) \rangle = \frac{1}{2} \langle T(s), T(s) \rangle' = 0$, we know

$$T'(s) = k(s)N(s).$$

So

$$N'(s) = J(T'(s)) = k(s)J(N(s)) = k(s)J(J(T(s))) = -k(x)T(s)$$

In summary, we have

$$\begin{bmatrix} T(s) \\ N(s) \end{bmatrix}' = \begin{bmatrix} 0 & k(s) \\ -k(s) & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \end{bmatrix}$$

Curvature in general parameter

Let $\alpha(t) : [0, b] \to \mathbb{R}^2$ be a regular curve.

Let s be the arc-length parametrization. So $\alpha(t) \to \alpha(t(s))$.

$T = \frac{d}{ds}\alpha(t(s)) = \alpha'(t(s))t'(s)$. So $1 = |T| = \alpha'(t(s))t'(s) \implies t'(s) = \frac{1}{\alpha'(t(s))}$.

$$\frac{d}{ds}T(t(s)) = \alpha''(t(s))(t'(s))^2 + \alpha'(t(s))t''(s)$$

Hence

$$k = \left\langle \frac{d}{ds}T, J(T) \right\rangle = \langle \alpha''(t')^2 + \alpha' t'', J(\alpha')t' \rangle$$
$$= (t')^3 \langle \alpha'', J(\alpha') \rangle = \frac{\det(\alpha', \alpha'')}{|\alpha'|^3}$$

Geometric meaning of signed curvature.

Example of a simple curve.

$\alpha(t) : (-\infty, \infty) \to \mathbb{R}^2$, given by

$$\alpha(t) = (t, t^3).$$

Then

$$\alpha'(t) = (1, 3t^2)$$
$$\alpha''(t) = (0, 6t)$$
$$k(t) = \frac{6t}{(1 + 9t^4)^{\frac{3}{2}}}$$

In summary,

- $|k|$ large, α rotates quickly.
- $|k|$ small, α is closed to a line.
- $k < 0$. α turning right, away from N.
- $k > 0$. α turning left, towards N.

Curvature measures the rate of turning of the unit tangent vectors.

Proposition. *Let $\alpha(s) : I \to \mathbb{R}^2$ be a regular curve parameterized by arc-length. Let $\theta(s)$ be the angle of $T(s)$ measured from x-axis. Then $k(s) = \theta'(s)$.*

Proof. Since $|T(s)| = 1$ and has angle θ from x-axis, we can write

$$T(s) = (\cos(\theta(s)), \sin\theta(s))$$

So

$$T'(s) = (-\sin(\theta(s)), \cos(\theta(s)))\theta'(s)$$

Since $N(s) = J(T(s)) = (-\sin(\theta(s)), \cos(\theta(s)))$, we have

$$k(s) = \langle T'(s), N(s)\rangle = \theta'(s)$$

□