Definition

Let $\mathbf{X} : U \to \mathbb{R}^3$ *be a regular surface patch, and let* $M = \mathbf{X}(U)$*. Let* $p \in M$ *be a point in the surface.* $p = \mathbf{X}(u_0^1, u_0^2)$ *for some* (u_0^1, u_0^2) *in* U*. Then the tangent space* $T_p(M)$ *of* M *at* p *is the vector space spanned by* $\mathbf{X}_1(u_0^1, u_0^2), \mathbf{X}_2(u_0^1, u_0^2)$*. Since* $\mathbf{X}_1, \mathbf{X}_2$ *are linearly independent,* $\dim(T_p(M)) = 2$*.*

Here $\mathbf{X}_1 = \dfrac{\partial \mathbf{X}}{\partial u^1}$, etc.

Tangent space is well-defined

Proposition

$T_p(M)$ is well defined. Namely, suppose $\phi : V \to U$ is a diffeomorphism, $V \subset \mathbb{R}^2$ with coordinates (v^1, v^2). Let $\mathbf{Y} = \mathbf{X} \circ \phi$. Then the vector space spanned by $\frac{\partial \mathbf{X}}{\partial u^1}$, $\frac{\partial \mathbf{X}}{\partial u^2}$, and the vector space spanned by $\frac{\partial \mathbf{Y}}{\partial v^1}$, $\frac{\partial \mathbf{Y}}{\partial v^2}$ are the same.

Tangent space consists of tangent vectors of curves on M

Lemma

Let $\mathbf{X} : U \to \mathbb{R}^3$ be a regular surface patch and let $M = \mathbf{X}(U)$. Let $\alpha(t)$ be a smooth curve in $\mathbb{R}^3$ such that $\alpha(t) \in M$ for all $t \in (a, b)$ passing through a point $p = \alpha(t_0)$ say. Then there is $\epsilon > 0$ and there is a unique smooth curve $\beta(t)$ in U with $t \in (t_0 - \epsilon, t_0 + \epsilon)$ such that $\alpha(t) = \mathbf{X}(\beta(t))$ in $(t_0 - \epsilon, t_0 + \epsilon)$.

Sketch of proof.

Let α, p as in the proposition and let $(u_0^1, u_0^2) \in U$ with $\mathbf{X}(u_0^1, u_0^2)$. By the lemma, we may assume that near p, the surface is a graph over xy-plane. Namely, there are open sets $\mathbf{u}_0 \in V \subset U$ and W and a diffeomorphism $\phi : W \to V$ with $\phi^{-1}(\mathbf{u}_0) = (x_0, y_0) \in W$ such that $\mathbf{Y}(x, y) = \mathbf{X} \circ \phi(x, y) = (x, y, f(x, y))$. Now $\alpha(t) \in \mathbf{X}(U)$ so $\alpha(t) = (x(t), y(t), f(x(t), y(t))) = \mathbf{Y}(x(t), y(t))$. Let $\beta(t) = \phi(x(t), y(t))$. Then $\mathbf{X}(\beta(t)) = \alpha(t)$.

□

Tangent space consists of tangent vectors of curves on M, cont.

Corollary

Let $\mathbf{X} : U \to \mathbb{R}^3$ be a regular surface patch, and let $M = \mathbf{X}(U)$. Let $p \in M$ be a point in the surface. Then $T_p(M)$ consists of the tangent vectors of smooth curves on M passing through p.

Definition

Let $\mathbf{X} : U \to \mathbb{R}^3$ *be a regular surface patch and let* $M = \mathbf{X}(U)$. *A nonzero vector* N *at a point* $p = \mathbf{X}(u^1, u^2) \in M$ *is called* *a normal vector* *of* M *at* p *if it is orthogonal to* $T_p(M)$. *A normal vector* $\mathbf{N}$ *at* p *is called* *a unit normal vector* *if* $\mathbf{N}$ *has unit length.*

Questions: How many normal vectors at a point are there? How many unit normal vectors?

Facts: (i) Suppose $\mathbf{X}(u, v)$ is a parametrization of a regular surface M. Then a normal of M at a point $\mathbf{X}(u, v)$ is given by $\mathbf{X}_u \times \mathbf{X}_v$. A unit normal is given by

$$\mathbf{N} = \frac{\mathbf{X}_u \times \mathbf{X}_v}{|\mathbf{X}_u \times \mathbf{X}_v|}.$$

(ii) Suppose M is the level set of a regular value of a smooth function f in an open set in $\mathbb{R}^3$. Then a unit normal of the surface is given by

Examples:

(i) Consider the sphere $\mathbb{S}^2(r) = \{x^2 + y^2 + z^2 = r^2\}$ which is the level set of $f(x, y, z) = x^2 + y^2 + z^2$ at the regular value r^2. Then

$$\mathbf{N} = (x, y, z)$$

if $r = 1$.

(ii) Consider the surface of revolution:

$$\mathbf{X}(u, v) = (f(u)\cos v, f(u)\sin v, g(u).$$

Then $X_u = (f'\cos v, f'\sin v, g')$, $\mathbf{X}_v = (-f(u)\sin v, f(u)\cos v, 0)$.

$$\begin{aligned}\mathbf{X}_u \times \mathbf{X}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ f'\cos v & f'\sin v & g' \\ -f\sin v & f\cos v & 0 \end{vmatrix} \\ &= -fg'\cos v\mathbf{i} - fg'\sin v\mathbf{j} + ff'\mathbf{k}\end{aligned}$$

$$|\mathbf{X}_u \times \mathbf{X}_v|^2 = \left(f^2(f')^2 + (g')^2\right).$$

$$\mathbf{X}(\theta, v) = (\cos\theta, \sin\theta, 0) + v(\sin\frac{1}{2}\theta\cos\theta, \sin\frac{1}{2}\theta\sin\theta, \cos\frac{1}{2}\theta)$$
$$= \mathbf{a}(\theta) + v\mathbf{w}(\theta)$$

$-\pi < \theta < \pi$, $-\frac{1}{2} < v < \frac{1}{2}$. When $\theta = \pi$, $\mathbf{w}(\theta) = (-1, 0, 0)$. When $\theta = -\pi$, then $\mathbf{w}(\theta) = (1, 0, 0)$. Now

$$\mathbf{X}_v = (\sin\frac{1}{2}\theta\cos\theta, \sin\frac{1}{2}\theta\sin\theta, \cos\frac{1}{2}\theta)$$
$$\mathbf{X}_\theta = (-\sin\theta, \cos\theta, 0) + v\mathbf{w}'(\theta)$$
$$\therefore \mathbf{X}_\theta(\theta, 0) = (-\sin\theta, \cos\theta, 0)$$

At $(\theta, 0)$

$$\mathbf{N} = \frac{\mathbf{X}_\theta \wedge \mathbf{X}_v}{|\mathbf{X}_\theta \wedge \mathbf{X}_v|} = (\cos\theta \cos\frac{1}{2}\theta, \sin\theta\cos\frac{1}{2}\theta, \sin\frac{1}{2}\theta).$$

$$\therefore \lim_{\theta \to -\pi} \mathbf{N}(\theta, 0) = (0, 0, -1)$$

$$\lim_{\theta \to \pi} \mathbf{N}(\theta, 0) = (0, 0, 1).$$

On the other hand, $\mathbf{X}(\pi, 0) = (-1, 0, 0) = \mathbf{X}(-\pi, 0)$
Hence the Mobiüs strip has no continuously defined unit normal vector field.

First fundamental form

Definition

Let $\mathbf{X} : U \to \mathbb{R}^3$ be a regular surface patch, and let $M = \mathbf{X}(U)$. Let $p \in M$ be a point in the surface.L The first fundamental form g of M at p is the inner product at each $T_p(M)$ given by $g(\mathbf{v}, \mathbf{w}) = \langle \mathbf{v}, \mathbf{w} \rangle$. The first fundamental form of M is the inner product given by $g(\mathbf{v}, \mathbf{w}) = \langle \mathbf{v}, \mathbf{w} \rangle$ on every $T_p(M)$ for with $p \in M$.

Sometimes $g(\mathbf{v}, \mathbf{w})$ is written as $\mathrm{I}(\mathbf{v}, \mathbf{w})$.

Coefficients of the 1st fundamental form

Let $\mathbf{X} : U \to V \subset M$ be a coordinate parametrization. The *coefficients of the first fundamental form g* with respect to the parametrization are defined as:

$$\begin{cases} E = & g(\mathbf{X}_u, \mathbf{X}_u) = \langle \mathbf{X}_u, \mathbf{X}_u \rangle; \\ F = & g(\mathbf{X}_u, \mathbf{X}_v) = \langle \mathbf{X}_u, \mathbf{X}_v \rangle; \\ G = & g(\mathbf{X}_u, \mathbf{X}_u) = \langle \mathbf{X}_v, \mathbf{X}_v \rangle. \end{cases}$$

if (u, v) denotes points in U.
If we use (u^1, u^2) instead of (u, v) and let $\mathbf{X}_i = \frac{\partial \mathbf{X}}{\partial u^i}$, then we also denote coefficients of the first fundamental form g as

$$g_{ij} = \langle \mathbf{X}_i, \mathbf{X}_j \rangle.$$

Length of a curve

Suppose $\alpha(t) = (x(t), y(t), z(t))$ is a smooth curve on M, $a \leq t \leq b$ such that $\alpha(t) = \mathbf{X}((u(t), v(t))$ in local coordinates.

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Then the length of α is given by

$$\begin{aligned}\ell &= \int_a^b |\alpha'|(t)dt \\ &= \int_a^b \left(E(\alpha(t))(\frac{du}{dt})^2 + 2F(\alpha(t))\frac{du}{dt}\frac{dv}{dt} + G(\alpha(t))(\frac{dv}{dt})^2 \right)^{\frac{1}{2}} dt.\end{aligned}$$

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If we use (u^1, u^2) instead of (u, v) and $\mathbf{X}_i = \frac{\partial \mathbf{X}}{\partial u^i}$,

$$\ell = \int_a^b \left(\sum_{i,j=1}^{2} g_{ij} \frac{du^i}{dt}\frac{du^j}{dt} \right)^{\frac{1}{2}} dt.$$

Length of a curve, cont.

So sometimes, the first fundamental form is written symbolically as

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2,$$

or

$$g = \sum_{i,j=1}^{2} g_{ij} du^i du^j.$$

Area of a region

Let $\mathbf{X} : U \to M$ be a parametrization of a regular surface. Let R be a closed and bounded region in $\mathbf{X}(U)$. Let $V = \mathbf{X}^{-1}(R)$. The area of R is given by

$$A(R) = \iint_V |\mathbf{X}_u \times \mathbf{X}_v| dudv = \iint_V \sqrt{EG - F^2}$$

where E, F, G are the coefficients of the first fundamental form w.r.t. this parametrization. It is well-defined: $A(R)$ is independent of parametrization.

Examples

Graphs: Let $M = \{(x, y, z) |\ z = f(x, y), (x, y) \in U \subset \mathbb{R}^2\}$. It is parametrized by $\mathbf{X}(u, v) = (u, v, f(u, v))$. Hence

$$E = 1 + f_u^2, F = f_u f_v, G = 1 + f_v^2.$$

The surface area of $\mathbf{X}(U)$ is given by

$$\begin{aligned} A &= \iint_U \sqrt{(1 + f_u^2)(1 + f_v^2) - f_u^2 f_v^2} dudv \\ &= \iint_U \sqrt{1 + f_u^2 + f_v^2} dudv \end{aligned}$$

Sphere: $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 | \ x^2 + y^2 + z^2 = 1\}$.
$\mathbb{S}^2$ can be covered by the following family of coordinate charts.
(i) One of them is $\mathbf{X}(x, y) = (x, y, \sqrt{1 - (x^2 + y^2)})$, $(x, y) \in D$ which is the unit disk in $\mathbb{R}^2$. This is graph. So the coefficients of the first fundamental form can be computed as before.
(ii) (Spherical coordinates) One of them is:

$$\mathbf{X}(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

with $\{(\theta, \varphi) | \ 0 < \theta < \pi, 0 < \varphi < 2\pi\}$.

$$\mathbf{X}_\theta = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta)$$

$$\mathbf{X}_\varphi = (-\cos\theta\sin\varphi, \cos\theta\cos\varphi, 0)$$

So $E = 1; F = 0; G = \cos^2\theta$.

(iii) (Stereographic projection) The unit sphere M is considered as the set $\{x^2 + y^2 + (z-1)^2 = 1\}$, parametrized by

$$\mathbf{X}(u,v) = \left(\frac{4u}{u^2+v^2+4}, \frac{4v}{u^2+v^2+4}, \frac{2(u^2+v^2)}{u^2+v^2+4}\right)$$

The first fundamental form is:

$$E = G = \frac{1}{1 + \frac{1}{4}(u^2+v^2)^2}; F = 0.$$