MATH4030 Assignment 2

Solution of (1)

We can write out $\boldsymbol{X}(s,v)$ as following. Suppose

$$\alpha(s) = (\cos s, \sin s, 0), \quad s \in [0, 2\pi]$$

Then

$$\boldsymbol{X}(s,v) = (\cos s, \sin s, 0) + v(-\sin s, \cos s, 1) = (\cos s - v\sin s, \sin s + v\cos s, v)$$

Note that

$$(\cos s - v\sin s)^2 + (\sin s + v\cos s)^2 - v^2 = 1 + v^2 - v^2 = 1.$$

So $\boldsymbol{X}$ is part of the hyperboloid $x^2 + y^2 - z^2 = 1$.

$\boldsymbol{X}$ is surjective since for any (x,y,z) such that $x^2+y^2-z^2 = 1$, we choose $v = z$. Now we can choose a unique $\theta \in [0, 2\pi)$ such that

$$\cos\theta = \frac{1}{\sqrt{1+v^2}}, \quad \sin\theta = \frac{v}{\sqrt{1+v^2}}.$$

Note that

$$x^2 + y^2 = 1 + v^2,$$

we can also choose a unique $s \in [0, 2\pi)$ such that

$$\cos(s+\theta) = \frac{x}{\sqrt{1+v^2}}, \quad \sin(s+\theta) = \frac{y}{\sqrt{1+v^2}}.$$

In summary, we will get

$$x = \sqrt{1+v^2}\cos(s+\theta) = \sqrt{1+v^2}(\cos s\cos\theta - \sin s\sin\theta) = \cos s - v\sin s,$$
$$y = \sqrt{1+v^2}\sin(s+\theta) = \sqrt{1+v^2}(\sin s\cos\theta + \cos s\sin\theta) = \sin s + v\cos s.$$

For such s, v, we have $\boldsymbol{X}(s,v) = (x,y,z)$.

$\boldsymbol{X}$ is not injective since we know

$$\boldsymbol{X}(0,v) = \boldsymbol{X}(2\pi, v).$$

$\boldsymbol{X}$ has rank 2 for $s \in (0, 2\pi)$ since

$$\boldsymbol{X}_s = \alpha'(s) + v\alpha''(s) \quad \text{(a vector on } x\text{-}y \text{ plane)}$$
$$\boldsymbol{X}_v = \alpha'(s) + e_3 \quad \text{(a vector not on } x\text{-}y \text{ plane).}$$

So $\boldsymbol{X}_s, \boldsymbol{X}_v$ are linearly independent. Hence $\boldsymbol{X}$ has rank 2.

Solution of (2)

After rotation along x-axis, we know the catenoid is the set of all points satisfying

$$y^2 + z^2 = \cosh^2 x.$$

So one of the possible parametrization is

$$\boldsymbol{X}(u, v) = (u, \cosh u \cos v, \cosh u, \sin v)$$

Hence

$$\begin{aligned}
\boldsymbol{X}_u &= (1, \sinh u \cos v, \sinh u, \sin v) \\
\boldsymbol{X}_v &= (0, -\cosh u \sin v, \cosh u \cos v) \\
\langle \boldsymbol{X}_u, \boldsymbol{X}_u \rangle &= 1 + \sinh^2 u = \cosh^2 u \\
\langle \boldsymbol{X}_u, \boldsymbol{X}_v \rangle &= 0 \\
\langle \boldsymbol{X}_v, \boldsymbol{X}_v \rangle &= \cosh^2 u
\end{aligned}$$

So the first fundamental form is

$$\mathrm{I} = \cosh^2 u du^2 + \cosh^2 u dv^2 = \cosh^2 u \left(du^2 + dv^2\right).$$

Solution of (3)

We need to check the following things.
$\boldsymbol{X}$ is smooth. This is easy to see since each coordinate function is smooth.
$\boldsymbol{X}$ has rank 2. We compute

$$\begin{aligned}
\boldsymbol{X}_u &= (1 - u^2 + v^2, 2uv, 2u) \\
\boldsymbol{X}_v &= (2uv, 1 - v^2 + u^2, -2v) \\
\boldsymbol{X}_u \times \boldsymbol{X}_v &= (-2u - 2uv^2 - 2u^3, 2v + 2u^2 v + 2v^3, 1 - u^4 - v^4 - 2u^2 v^2) \\
&= (-2u(1 + u^2 + v^2), 2v(1 + u^2 + v^2), 1 - (u^2 + v^2)^2)
\end{aligned}$$

So clearly we have

$$\begin{aligned}
|\boldsymbol{X}_u \times \boldsymbol{X}_v|^2 &= 4(1 + u^2 + v^2)^2 (u^2 + v^2) + (1 - (u^2 + v^2)^2)^2 > 0 \\
&= (1 + u^2 + v^2)^4 > 0.
\end{aligned}$$

Hence, $\boldsymbol{X}_u, \boldsymbol{X}_v$ are linearly independent, which shows $\boldsymbol{X}$ has rank 2.
$\boldsymbol{X}$ is injective when $u^2 + v^2 < 3$. Suppose we have $\boldsymbol{X}(u_1, v_1) = \boldsymbol{X}(u_2, v_2)$. At first, we assume $u_1 \neq u_2$. Now we have

$$u_1 - \frac{u_1^3}{3} + u_1 v_1^2 = u_2 - \frac{u_2^3}{3} + u_2 v_2^2, \tag{1}$$

$$u_1^2 - v_1^2 = u_2^2 - v_2^2 \tag{2}$$

So from (1), we have

$$\begin{aligned}
0 &= u_1 - u_2 - \frac{1}{3}(u_1^3 - u_2^3) + u_1 v_1^2 - u_2 v_2^2 \\
&= u_1 - u_2 - \frac{1}{3}(u_1 - u_2)(u_1^2 + u_1 u_2 + u_2^2) + (u_1 - u_2)v_1^2 + u_2(v_1^2 - v_2^2) \\
&= u_1 - u_2 - \frac{1}{3}(u_1 - u_2)(u_1^2 + u_1 u_2 + u_2^2) + (u_1 - u_2)v_1^2 + u_2(u_1^2 - u_2^2) (\text{by } (2)) \\
&= (u_1 - u_2)\left(1 - \frac{1}{3}u_1^2 + \frac{2}{3}u_1 u_2 + \frac{2}{3}u_2^2 + v_1^2\right)
\end{aligned}$$

Since $u_1 \neq u_2$, we have

$$\begin{aligned}
0 &= 1 - \frac{1}{3}u_1^2 + \frac{2}{3}u_1 u_2 + \frac{2}{3}u_2^2 + v_1^2 \\
&\geq 1 - \frac{1}{3}u_1^2 - \frac{1}{3}u_1^2 + \frac{1}{3}u_2^2 + v_1^2 \quad (2u_1 u_2 \geq -u_1^2 - u_2^2) \\
&= 1 - \frac{1}{3}u_1^2 - \frac{1}{3}v_1^2 + \frac{1}{3}v_2^2 + v_1^2 \quad (\text{by } (2)) \\
&> \frac{1}{3}v_2^2 + v_1^2 \quad (\text{by } u_1^2 + v_1^2 < 3) \\
&\geq 0
\end{aligned}$$

A contradiction. So we need to have $u_1 = u_2$. By the symmetric of u, v, we know $v_1 = v_2$. This is a contradiction with the assumption of u_1, u_2, v_1, v_2.

So $\boldsymbol{X}$ is injective when $u^2 + v^2 < 3$.

But we know $\boldsymbol{X}(\sqrt{3}, 0) = (0, 0, 3) = \boldsymbol{X}(-\sqrt{3}, 0)$.

For the unit normal, we have

$$\begin{aligned}
\boldsymbol{N} = \frac{\boldsymbol{X}_u \times \boldsymbol{X}_v}{|\boldsymbol{X}_u \times \boldsymbol{X}_v|} &= \frac{(-2u(1+u^2+v^2), 2v(1+u^2+v^2), 1-(u^2+v^2)^2)}{(1+u^2+v^2)^2} \\
&= \left(\frac{-2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{1-u^2-v^2}{1+u^2+v^2}\right).
\end{aligned}$$

Solution of (4)

We consider

$$\boldsymbol{X}(u, v) = \left(\frac{4u}{u^2+v^2+4}, \frac{4v}{u^2+v^2+4}, \frac{2(u^2+v^2)}{u^2+v^2+4}\right).$$

Then

$$\begin{aligned}
\boldsymbol{X}_u &= \frac{4}{(u^2+v^2+4)^2}\left(-u^2+v^2+4, -2uv, 4u\right), \\
\boldsymbol{X}_v &= \frac{4}{(u^2+v^2+4)^2}(-2uv, u^2-v^2+4, 4v), \\
E &= \langle \boldsymbol{X}_u, \boldsymbol{X}_u \rangle = \frac{16}{(u^2+v^2+4)^2}, \\
F &= \langle \boldsymbol{X}_u, \boldsymbol{X}_v \rangle = 0, \\
G &= \langle \boldsymbol{X}_v, \boldsymbol{X}_v \rangle = \frac{16}{(u^2+v^2+4)^2}.
\end{aligned}$$

Solution of (5)

The length of α is given by

$$\begin{aligned}\text{Length}(\alpha) &= \int_a^b |\alpha'(t)|\,dt = \int_a^b \sqrt{\cos^2 t \cos^2 u_0 + \cos^2 t \sin^2 u_0 + \sin^2 t} \\ &= \int_a^b 1 dt = b - a.\end{aligned}$$

If β is another curve jointing $\alpha(a), \alpha(b)$, we know

$$\begin{aligned}\beta'(t) &= \boldsymbol{X}_u u'(t) + \boldsymbol{X}_v u'(t) \\ &= (-\sin v \sin u, \sin v \cos u, 0)u'(t) + (\cos v \cos u, \cos v \sin u, -\sin v)v'(t).\end{aligned}$$

Note that $\langle \boldsymbol{X}_u, \boldsymbol{X}_v \rangle = 0, \langle \boldsymbol{X}_v, \boldsymbol{X}_v \rangle = 1$, we have

$$\begin{aligned}\text{Length}(\beta) &= \int_a^b |\beta'(t)|\,dt = \int_a^b \sqrt{\langle \boldsymbol{X}_u, \boldsymbol{X}_u \rangle u'(t)^2 + \langle \boldsymbol{X}_v, \boldsymbol{X}_v \rangle v'(t)^2} dt \\ &\geq \int_a^b |v'(t)|\,dt \geq \int_a^b v'(t)dt = v(b) - v(a)\end{aligned}$$

Then by $\beta(a) = X(u_0, a)$, we know $v(a) = a$. Similarly, we have $v(b) = b$. Hence

$$\text{Length}(\beta) \geq b - a.$$

So

$$l(\beta) \geq l(\alpha).$$

Solution of (6)

We have the following calculation.

$$\begin{aligned}\boldsymbol{X}_u &= (-r \sin u \cos v, -r \sin u \sin v, r \cos u). \\ \boldsymbol{X}_v &= (-(a + r\cos u)\sin v, (a + r \cos u)\cos v, 0), \\ E &= \langle \boldsymbol{X}_u, \boldsymbol{X}_u \rangle = r^2(\sin^2 u \cos^2 v + \sin^2 u \sin^2 v + \cos^2 u) = r^2, \\ F &= \langle \boldsymbol{X}_u, \boldsymbol{X}_v \rangle = ar \sin u \sin v \cos v - ar \sin u \sin v \cos v \\ &\quad + r^2 \cos u \sin u \sin v \cos v - r^2 \cos u \sin u \sin v \cos v = 0, \\ G &= \langle \boldsymbol{X}_v, \boldsymbol{X}_v \rangle = (a + r\cos u)^2.\end{aligned}$$

So the area of this torus is given by

$$\begin{aligned}\text{Area} &= \int_0^{2\pi} \int_0^{2\pi} \sqrt{EG - F^2} dudv \\ &= \int_0^{2\pi} \int_0^{2\pi} r(a + r \cos u) dudv = \int_0^{2\pi} 2\pi ar dv = 4\pi^2 ar.\end{aligned}$$