MATH4030 Assignment 1

Solution of (1)

(a). We compute

$$\alpha'(t) = \left(\cos t, -\sin t + \frac{\cos\frac{t}{2}}{2\sin\frac{t}{2}} + \frac{\sin\frac{t}{2}}{2\cos\frac{t}{2}}\right) = \left(\cos t, -\sin t + \frac{1}{\sin t}\right)$$
$$|\alpha'(t)|^2 = \cos^2 t + \sin^2 t - 2 + \frac{1}{\sin^2 t} = \frac{1}{\sin^2 t} - 1 = \cot^2 t$$

Hence $|\alpha'(t)| > 0$ if and only if $t = \frac{\pi}{2}$. So α is regular except at $t = \frac{\pi}{2}$.
(b). Consider the line $\beta(s) := \alpha(t) + s\alpha'(t)$, which passes through $\alpha(t)$ with direction $\alpha'(t)$. It's easy to see when $s = -\tan t$, $\beta(s)$ will locate on y-axis. In this case, we have

$$|\beta(0) - \beta(s)| = |\tan t| \left|\left(\cos t, -\sin t + \frac{1}{\sin t}\right)\right| = |\tan t|\,|\cot t| = 1$$

for any $t \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$.

Solution of (2)

First part. $\Longrightarrow$. If $\langle T, \boldsymbol{u}\rangle = C$ is constant, then take derivative with respect to s will give

$$\langle \kappa N, \boldsymbol{u}\rangle = 0$$

So we have $\langle N, \boldsymbol{u}\rangle = 0$ since we assume $\kappa > 0$. Take derivative of $\langle N, \boldsymbol{u}\rangle = 0$ again will give

$$\langle -\kappa T + \tau B, \boldsymbol{u}\rangle = 0 \implies \frac{\kappa}{\tau} = \frac{\langle B, \boldsymbol{u}\rangle}{\langle T, \boldsymbol{u}\rangle}.$$

Note that $|\boldsymbol{u}|^2 = \langle T, \boldsymbol{u}\rangle^2 + \langle N, \boldsymbol{u}\rangle^2 + \langle B, \boldsymbol{u}\rangle^2 = C^2 + \langle B, \boldsymbol{u}\rangle^2$ since $\{T, N, B\}$ is a orthonormal basis. So $\langle B, \boldsymbol{u}\rangle$ is a constant. This means $\frac{\kappa}{\tau}$ is a constant.

Second part. $\Longleftarrow$. If $\frac{\kappa}{\tau}$ is constant, we choose $\boldsymbol{u} = \frac{\kappa}{\tau}B + T$. Note that $\boldsymbol{u}$ is a constant vector since

$$\boldsymbol{u}' = \frac{\kappa}{\tau}B' + T' = \frac{\kappa}{\tau}(-\tau N) + \kappa N = 0$$

Further more, we know

$$\langle T, \boldsymbol{u}\rangle = 1$$

is a constant.

Solution of (3)

First part. If α on a sphere, suppose the origin of this sphere is $(0,0,0)$ since the conclusion is translation invariant. Then we know $|\alpha|^2 = R^2$ is a constant. Taking differential twice will give

$$\begin{aligned} \langle \alpha(s), T(s) \rangle = 0 \implies \\ \langle T(s), T(s) \rangle + \langle \alpha(s), \kappa N(s) \rangle = 0 \\ \implies \kappa \langle \alpha, N \rangle = -1 \end{aligned}$$

Taking differential again of above will give

$$\begin{aligned} 0 = \kappa' \langle \alpha, N \rangle + \kappa \langle T, N \rangle + \kappa \langle \alpha, -\kappa T + \tau B \rangle \implies \\ \langle \alpha, B \rangle = \frac{\kappa'}{\kappa\tau} \langle \alpha, N \rangle = -\frac{\kappa'}{\kappa^2 \tau} \end{aligned}$$

where we've used $\langle T, N \rangle = 0$ and $\langle \alpha, N \rangle = -\frac{1}{\kappa}, \langle \alpha, T \rangle = 0$. By the definition of ρ, λ,we have

$$\alpha = \langle \alpha, T \rangle T + \langle \alpha, N \rangle N + \langle \alpha, B \rangle B = -\rho N - \rho' \lambda B.$$

So $\rho^2 + (\rho')^2 \lambda^2 = |\alpha|^2 = R^2$ is a constant.

Second part. We assume $\rho^2 + (\rho')^2 \lambda^2 = C$ is a constant. Now we choose

$$\beta = \alpha + \rho N + \rho' \lambda B$$

We want to show β is a constant vector. So after taking derivative, we have

$$\begin{aligned} \beta' &= T + \rho' N + \rho(-\kappa T + \tau B) + \rho'' \lambda B + \rho' \lambda' B + \rho' \lambda(-\tau N) \\ &= T + \rho' N - T + \rho \tau B + \rho'' \lambda B + \rho' \lambda' B - \rho' N \\ &= (\frac{\rho}{\lambda} + \rho'' \lambda + \rho' \lambda') B \end{aligned}$$

If we different the condition $\rho^2 + (\rho')^2 \lambda^2 = C$,we have

$$0 = \rho \rho' + \rho' \rho'' \lambda^2 + (\rho')^2 \lambda \lambda'$$

Since $\rho' \neq 0$(from $\kappa' \neq 0$), we know

$$\frac{\rho}{\lambda} + \rho'' \lambda + \rho' \lambda' = 0.$$

Hence $\beta' = 0$. This shows β is a constant vector. So we know α is on a sphere since $|\alpha - \beta| = |\rho N + \rho' \lambda B| = \sqrt{\rho^2 + (\rho')^2 \lambda^2}$ is a constant.

Solution of (4)

Suppose $\alpha(s_1^i), \alpha(s_2^i), \alpha(s_3^i)$ are collinear with $s_3^i > s_2^i > s_1^i$ tend to s_0. Let L^i be the line $\alpha(s_1^i), \alpha(s_2^i), \alpha(s_3^i)$ located and P^i be the subspace of $\mathbb{R}^3$ that is orthogonal to L. So for any orthonormal basis $\{e_1^i, e_2^i\}$ of P^i, we consider the function $f_j^i(s) = \langle e_j^i, \alpha(s)\rangle$. Note that $f_j^i(s_1^i) = f_j^i(s_2^i) = f_j^i(s_3^i)$ for any i and $j = 1, 2$, we can apply mean value theorem to get

$$\left\langle e_j^i, T(\xi_1^i)\right\rangle = \left\langle e_j^i, T(\xi_2^i)\right\rangle = 0$$

for some $\xi_1^i \in (s_1^i, s_2^i), \xi_2^i \in (s_2^i, s_3^i)$. We apply mean value theorem again to get

$$\left\langle e_j^i, \kappa(\eta^i)N(\eta^i)\right\rangle = 0$$

for some $\eta^i \in (\xi_1^i, \xi_2^i)$.

Without loss generality, we assume e_1^i, e_2^i tend to the vectors e_1, e_2 by compactness. So the

Now we can take limit $i \to \infty$ for above three identities to get

$$\langle e_j, T(s_0)\rangle = 0 \text{ and } \langle e_j, N(s_0)\rangle = 0$$

since $\kappa > 0$. This is impossible since $T(s_0), N(s_0)$ will be orthogonal to a plane spanned by e_1, e_2, which shows $T(s_0), N(s_0)$ are collinear.

Now we will show that for any $s_1 < s_2 < s_3$ that $s_1, s_2, s_3 \to s_0$, the unique plane contains $\alpha(s_1), \alpha(s_2), \alpha(s_3)$ will approach to the plane spanned by $T(s_0), N(s_0)$. Again, if this is not true, we can find a sequence of s_1^i, s_2^i, s_3^i such that above does not hold. This means if we let P^i be the plane containing $\alpha(s_1^i), \alpha(s_2^i), \alpha(s_3^i)$, let e^i be the unit normal of P^i, then the e^i will not converge to the $B(s_0)$ or $-B(s_0)$ since $B(s_0)$ is the normal of plane spanned by $T(s_0), N(s_0)$.

Again, since

$$\left\langle e^i, \alpha(s_1^i) - \alpha(s_2^i)\right\rangle = 0 = \left\langle e^i, \alpha(s_2^i) - \alpha(s_3^i)\right\rangle$$

we can apply mean value theorem to find $\xi_1^i \in (s_1^i, s_2^i), \xi_2^i \in (s_2^i, s_3^i), \eta \in (\xi_1^i, \xi_2^i)$ such that

$$\left\langle e^i, T(\xi_1^i)\right\rangle = \left\langle e^i, T(\xi_2^i)\right\rangle = 0$$
$$\left\langle e^i, \kappa(\eta^i)N(\eta^i)\right\rangle = 0$$

Again, we suppose $e^i \to e$ by compactness upto a subsequence, which we know $e \neq B(s_0)$ or $e \neq -(B(s_0))$ by our assumption. Moreover, we know $|e| = 1$.

After taking $i \to \infty$, we have

$$\langle e, T(s_0)\rangle = \langle e, N(s_0)\rangle = 0$$

So actually e is a normal vector of P spanned by $T(s_0), N(s_0)$. This is a contradiction with e cannot be $B(s_0)$ or $-B(s_0)$.

So we know the plane containing $\alpha(s_1), \alpha(s_2), \alpha(s_3)$ actually converges to the plane spanned by $T(s_0), N(s_0)$.

Solution of (5)

For circular helix, if $\alpha(s) = \left(a\cos\frac{s}{\sqrt{a^2+b^2}}, a\sin\frac{s}{\sqrt{a^2+b^2}}, \frac{bs}{c}\right)$ with $a > 0$, then it has $\kappa = \frac{a}{a^2+b^2}, \tau = -\frac{b}{a^2+b^2}$. So for any given $\kappa_0 > 0, \tau_0 \neq 0$ to be two constants, we can choose some $C \in \mathbb{R}$ and $\theta \in [-\pi, \pi]$ such that $\kappa_0 = C\cos\theta, \tau_0 = -C\sin\theta$. This is possible since we only need to choose $C = \sqrt{\kappa_0^2 + \tau_0^2}$ and $\tan\theta = -\frac{\tau_0}{\kappa_0}$ with $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ since $\kappa_0 > 0$. Now we choose $a = \frac{\cos\theta}{C}$, $b = \frac{\sin\theta}{C}$. Clearly we have $a^2 + b^2 = \frac{1}{C^2}$. Hence

$$\kappa_0 = C\cos\theta = C^2 a = \frac{a}{a^2 + b^2}, \quad \tau_0 = -C\sin\theta = -C^2 b = -\frac{b}{a^2 + b^2}.$$

with $a > 0$. Hence the circular helix defined by

$$\alpha(s) = \left(a\cos\frac{s}{\sqrt{a^2 + b^2}}, a\sin\frac{s}{\sqrt{a^2 + b^2}}, \frac{bs}{c}\right)$$

will have the constant $\kappa = \kappa_0, \tau = \tau_0$. Now by the uniqueness result, for any $\kappa = \kappa_0, \tau = \tau_0$, the curve should be a circular helix define here. Since we can choose κ_0, τ_0 arbitrary, we can get when κ, τ is constant, then the curve α is a circular helix.