- A *(parametrized smooth) curve* $\alpha(t)$ is a smooth map

$$\alpha : I \subset \mathbb{R} \to \mathbb{R}^3$$

  from an interval I in $\mathbb{R}$ into $\mathbb{R}^3$ so that α is smooth. α is said to be *regular* if $\alpha' \neq 0$.
- Let $\alpha : (a, b) \to \mathbb{R}^3$ is a curve. Let $f : (c, d) \to (a, b)$ with $t = f(\sigma)$ such that $f' > 0$, then $\alpha(f(\sigma)) : (c, d) \to \mathbb{R}^3$ is said to be a *reparametrization* of α.

Arc-length

- Let α be a regular curve defined on $[a, b]$ and let $t_0 \in [a, b]$, the *arc-length* is defined as:

$$s(t) = \int_{t_0}^{t} |\alpha'(u)| du.$$

- If $s(a) = -L_1, s(b) = L_2$, then $\alpha(s) = \alpha(s(t))$ is a reparametrization of α and $\alpha(s)$ is said to be parametrized by arc-length.

■ $\alpha = \alpha(t)$ is parametrized by arc-length, that is t 'represents' arc-length from a fixed point iff $|\alpha'| = 1$.

Proof.

Suppose $|\alpha'| = 1$, then $s(t) = t - t_0$, so t 'represents' arc-length. Suppose t 'represents' arc-length in the sense that $t = s(t) + c$ with c is a constant. Then $s'(t) = 1$. Hence $|\alpha'| = 1$. □

The Frenet formula: Let $\alpha(s)$ be the regular curve parametrized by arc length s.
Let $\vec{T} = \alpha'$ (tangent). Then

$$\kappa(s) := |T'|(s) \quad \text{(curvature)};$$
$$N(s) := \frac{1}{k(s)} T'(s) \quad (\text{normal, if } \kappa > 0);$$
$$B(s) := T(s) \times N(s) \quad (\text{binormal, if } \kappa > 0).$$

Theorem

(Frenet formula) Let α be a regular curve parametrized by arc-length with curvature $\kappa > 0$. Then $B' = -\tau N$ for some τ. Moreover,

$$\begin{pmatrix} T \\ N \\ B \end{pmatrix}' = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

τ is called the torsion of α.

Proof.

$B = T \times N$. Since B has length 1, so $B' \perp B$. So $B' = aT - \tau N$.
But $\langle B', T\rangle = \langle T' \times N + T \times N', T\rangle = 0$. Hence $a = 0$.
Now, $N' = aT + bB$ because $N' \perp N$.
$a = \langle N', T\rangle = -\langle N, T'\rangle = -\kappa$. Similarly, one can prove that
$b = \tau$.

□

Proposition

Let $\alpha(t)$ be a regular curve with nonzero curvature. Then the curvature and torsion are given by:

$$\begin{cases} \kappa = & \frac{|\alpha' \times \alpha''|}{|\alpha'|^3} \\ \tau = & \frac{<\alpha' \times \alpha'', \alpha'''>}{|\alpha' \times \alpha''|^2}. \end{cases}$$

Here $'$ always means differentiation with respect to t.

Proof

:

Let $\alpha(t)$ be a regular curve with nonzero curvature. Then

$$\alpha' = |\alpha'|T,$$

$$\alpha'' = \kappa|\alpha'|^2 N + |\alpha'|^{-1} < \alpha', \alpha'' > T. \tag{1}$$

Hence

$$< \alpha'', \alpha'' >= \kappa^2|\alpha'|^4 + |\alpha'|^{-2} < \alpha', \alpha'' >^2,$$

and

$$\begin{aligned} \kappa^2 &= \frac{< \alpha'', \alpha'' >< \alpha', \alpha' > - < \alpha', \alpha'' >^2}{|\alpha'|^6} \\ &= \frac{|\alpha' \times \alpha''|^2}{|\alpha'|^6}. \end{aligned}$$

To compute τ, note that

$$\alpha''' = \kappa(-kT + \tau B)|\alpha'|^3 + f(t)T + g(t)N$$

for some function f and g. (Why?). So

$$\tau = \frac{1}{\kappa}\frac{< \alpha''', B >}{|\alpha'|^3}.$$

Use (1)

$$\begin{aligned} B &= T \times N \\ &= \frac{T \times \alpha''}{k|\alpha'|^2} \\ &= \frac{\alpha' \times \alpha''}{k|\alpha'|^3} \end{aligned}$$

Use the formula for k, we have

$$\tau = \frac{< \alpha' \times \alpha'', \alpha''' >}{|\alpha' \times \alpha''|^2}.$$

Let α be a regular curves in $\mathbb{R}^3$ parametrized by arc length.

- Suppose the curvature $\kappa \equiv 0$ if and only if α is a straight line.
  Proof: If $\kappa \equiv 0$, then $T' = 0$ and $\alpha' = T = \mathbf{a}$ is constant. So $\alpha = \mathbf{a}t + \mathbf{b}$ with $\mathbf{a}, \mathbf{b}$ being constant vectors. α is a straight line.
- Suppose the curvature $\kappa > 0$ and the torsion $\tau \equiv 0$ if and only if α is a plane curve.
  Proof: If it is a plane curve, then T, N always in a fixed plane. Hence B is constant and $B' = 0$. So $\tau \equiv 0$. If $\tau \equiv 0$, then $B' = 0$. That is B is a constant vector. $\langle \alpha(s) - \alpha(s_0), B \rangle' = 0$. Hence $\langle \alpha(s) - \alpha(s_0), B \rangle \equiv 0$, and α is plane curve.

- Suppose the curvature $\kappa = \kappa_0 > 0$ is a constant and $\tau \equiv 0$, then α is a circular arc with radius $1/\kappa_0$.
  Proof: May assume that α is in the xy-plane. Then $(\alpha + \frac{1}{\kappa_0}N)' = 0$. Hence $\alpha + \frac{1}{\kappa_0}N = \mathbf{a}$ is a constant and $|\alpha - \mathbf{a}| = \frac{1}{\kappa_0}$.
- Suppose the curvature $k > 0$ and the torsion $\tau \neq 0$ everywhere. α lies on s sphere if and only if $\rho^2 + (\rho')^2\lambda^2 =$constant, where $\rho = 1/k$ and $\lambda = 1/\tau$.
  Proof: Exercise.

- Suppose α is defined on $[a, b]$. Let $\mathbf{p} = \alpha(a)$ and $\mathbf{q} = \alpha(b)$. Then $L(\alpha) \geq |\mathbf{p} - \mathbf{q}|$. Equality holds if and only if α is the straight line from $\mathbf{p}$ to $\mathbf{q}$.
  Proof: We may assume that $|\alpha'| = 1$. Then

$$|x(b) - x(a)|^2 = (\int_a^b x' ds)^2 \leq (b - a) \int_a^b (x')^2 ds.$$

etc. So

$$|\mathbf{p} - \mathbf{q}|^2 \leq (b - a) \int_a^b |\alpha'|^2 ds = (b - a)^2 = L^2(\alpha).$$

Equality is true if and only if x', y', z' are proportional to s. Hence L is a straight line from $\mathbf{p}$ to $\mathbf{q}$.

- Suppose the curvature $k = k_0 > 0$ is a constant and $\tau = \tau_0$ is a constant. Then α is a circular helix.

Let $\alpha(t) = (a\cos t, a\sin t, bt)$. Then

$$\alpha'(t) = (-a\sin t, a\cos t, b)$$

Arc length from $\alpha(0)$, say is:

$$s(t) = \int_0^t |\alpha'(\sigma)|d\sigma = \int_0^t (a^2 + b^2)^{\frac{1}{2}} d\sigma = tc$$

where $c > 0$ with $a^2 + b^2 = c^2$
Hence $\alpha(s) = (a\cos\frac{s}{c}, a\sin\frac{s}{c}, b \cdot \frac{s}{c})$.

$$T(s) = (-\frac{a}{c}\sin\frac{s}{c}, \frac{a}{c}\cos\frac{s}{c}, \frac{b}{c}),$$

$$T'(s) = (-\frac{a}{c^2}\cos\frac{s}{c}, -\frac{a}{c^2}\sin\frac{s}{c}, 0),$$

Hence $\kappa = \frac{a}{c^2} = \frac{a}{a^2+b^2}$, and

$$N(s) = (-\cos\frac{s}{c}, -\sin\frac{s}{c}, 0); N'(s) = (\frac{1}{c}\sin\frac{s}{c}, -\frac{1}{c}\cos\frac{s}{c}, 0).$$

$$\begin{aligned}\tau &= - < N', B > \\ &= - < N', T \times N > \\ &= -\det\begin{pmatrix} -\frac{a}{c}\sin\frac{s}{c} & \frac{a}{c}\cos\frac{s}{c} & \frac{b}{c} \\ -\cos\frac{s}{c} & -\sin\frac{s}{c} & 0 \\ \frac{1}{c}\sin\frac{s}{c} & -\frac{1}{c}\cos\frac{s}{c} & 0 \end{pmatrix} = -\frac{b}{c^2} = -\frac{b}{a^2+b^2}.\end{aligned}$$