

Assignment 3, Due 21/10/2021 on or before 11:59 pm

Please upload your assignment to the Blackboard of this course

All surfaces are assumed to be orientable.

- (1) Let M_1, M_2 be two regular surfaces and let $F : M_1 \rightarrow M_2$ be a smooth map. Let $p \in M$ and $q = f(p) \in M_2$. Define the differential $dF : T_p(M_1) \rightarrow T_q(M_2)$ as follows: For any $\mathbf{v} \in T_p(M)$, let $\alpha(t)$ be a smooth curve on M_1 so that $\alpha(0) = p, \alpha'(0) = \mathbf{v}$. Define

$$dF_p(\mathbf{v}) = \frac{d}{dt}F(\alpha(t))|_{t=0}.$$

Prove that dF_p is well-defined and is linear.

(Consider the Gauss map $\mathbf{N} : M \rightarrow \mathbb{S}^2$. One can identify $T_p(M)$ with $T_{\mathbf{N}(p)}(\mathbb{S}^2)$. Then $\mathcal{S}_p = -d\mathbf{N}_p$.)

- (2) Consider the tractrix Let $\alpha : (0, \frac{\pi}{2}) \rightarrow xz$ -plane given by

$$\alpha(t) = \left(\sin t, 0, \cos t + \log \tan \frac{t}{2} \right).$$

Show that the Gaussian curvature of the surface of revolution obtained by rotating α about the z -axis is -1 . The surface is called the pseudosphere.

- (3) Let $M = \{(x, y, z) \mid z = x^2 + ky^2\}$, with $k > 0$. Show that $e_1 = (1, 0, 0)$ and $e_2 = (0, 1, 0)$ form a basis of $T_p(M)$ where $p = (0, 0, 0)$. Let \mathbf{N} be the unit normal of M pointing upward, i.e. $\langle \mathbf{N}, e_3 \rangle > 0$ where $e_3 = (0, 0, 1)$. Find the matrix of $\mathcal{S}_p : T_p(M) \rightarrow T_p(M)$ with respect to the ordered basis $\{e_1, e_2\}$. Find the principal curvatures of M at p .
- (4) (Gaussian curvature of an ellipsoid) Let M be a regular orientable surface with unit normal vector field \mathbf{N} . Let f be a smooth function on M which is nowhere zero. Let $p \in M$ and let \mathbf{v}_1 and \mathbf{v}_2 form an orthonormal basis for $T_p(M)$.

- (i) Prove that the Gaussian curvature of M at p is given by:

$$K = \frac{\langle d(f\mathbf{N})(\mathbf{v}_1) \times d(f\mathbf{N})(\mathbf{v}_2), \mathbf{N} \rangle}{f^2}.$$

Note $d(f\mathbf{N})(\mathbf{v})$ is defined as follow: let α be the curve on M with $\alpha(0) = p, \alpha'(0) = \mathbf{v}$, then

$$d(f\mathbf{N})(\mathbf{v}) = \frac{d}{dt}(f(\alpha(t))\mathbf{N}(\alpha(t)))|_{t=0}.$$

(ii) Let M be the ellipsoid

$$h(x, y, z) := \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Let f be the restriction of the function

$$\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} \right)^{\frac{1}{2}}.$$

Apply (i) to show that the Gaussian curvature is given by

$$K = \frac{1}{f^4 a^2 b^2 c^2}.$$

(Hint: We may take $\mathbf{N} = \frac{\nabla h}{|\nabla h|}$. Note that $|\nabla h| = 2f$ and so $d(f\mathbf{N})(\mathbf{v}) = \left(\frac{v_1}{a^2}, \frac{v_2}{b^2}, \frac{v_3}{c^2} \right)$ if $\mathbf{v} = (v_1, v_2, v_3)$.)

- (5) Let $\lambda_1, \dots, \lambda_m$ be the normal curvatures at p in a regular surface along directions making angles $0, \frac{2\pi}{m}, \dots, \frac{(m-1)2\pi}{m}$ with a principal direction, $m > 2$. Prove that the mean curvature H at p is given by $mH = \lambda_1 + \dots + \lambda_m$.
- (6) Let C be a regular curve on a surface regular surface M with Gaussian curvature $K > 0$ at $p \in C$. Show that the curvature κ of C at p satisfies $\kappa \geq \min\{|k_1|, |k_2|\}$ where k_1 and k_2 are the principal curvatures of M at p . Is the result true if $K < 0$ at p ?