

**Assignment 1, Due 22/9/2021 on or before 11:59 pm**

**Please upload your assignment to the Blackboard of this course**

- (1) (The tractrix) Let  $\alpha : (0, \pi) \rightarrow \mathbb{R}^2$  be given by

$$\alpha(t) = \left( \sin t, \cos t + \log \tan \frac{t}{2} \right).$$

- (a) Prove that  $\alpha$  is regular except at  $t = \frac{\pi}{2}$ .  
(b) Prove that the length of the segment of the tangent of  $\alpha$  between the point of tangency and the  $y$ -axis is constantly 1.
- (2) A regular curve  $\alpha(s)$  parametrized by arc length is called a *cylindrical helix* if there is some constant vector  $\mathbf{u}$  such that  $\langle T, \mathbf{u} \rangle = \cos \theta_0$  is a constant. Prove that a regular curve  $\alpha$  parametrized by arc length with  $\kappa > 0$  is a cylindrical helix if and only if  $\kappa/\tau$  is constant.
- (3) Assume that  $\kappa(s) > 0$ ,  $\tau(s) \neq 0$  and  $\kappa'(s) \neq 0$  for all  $s$  for a regular curve  $\alpha(s)$  parametrized by arc length. Show that  $\alpha$  lies on a sphere if and only if

$$\rho^2 + (\rho')^2 \lambda^2 = \text{constant}.$$

where  $\rho = 1/\kappa(s)$ ,  $\lambda = 1/\tau$ .

(Hint: Necessity: Differentiate  $|\alpha|^2$  three times to obtain  $\alpha = -\rho N - \rho' \lambda B$ . Sufficiency: Show that  $\beta = \alpha + \rho N + \rho' \lambda B$  is constant.)

- (4) Let  $\alpha : (a, b) \rightarrow \mathbb{R}^3$  be a regular smooth curve parametrized by arc length so that the curvature  $\kappa(s) > 0$  everywhere. Let  $s_0 \in (a, b)$ . Show that for  $s_3 > s_2 > s_1$  sufficiently close to  $s_0$ ,  $\alpha(s_1), \alpha(s_2), \alpha(s_3)$  do not lie on a line. Prove also that as  $s_i \rightarrow s_0$ , the unique plane containing  $\alpha(s_1), \alpha(s_2), \alpha(s_3)$  will approach the plane spanned by  $T(s_0), N(s_0)$ .
- (5) Show that a regular curve is a circular helix if  $\kappa > 0$  is a constant and  $\tau \neq 0$  is a constant.