THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4010 Functional Analysis 2021-22 Term 1
Solution to Homework 4

1. If X and Y are Banach spaces and $T_n : X \to Y$, $n = 1, 2, \dots$ a sequence of bounded linear operators, show that the following statements are equivalent:

   (a) the sequence $(\|T_n\|)$ is bounded,

   (b) the sequence $(\|T_n x\|)$ is bounded for every $x \in X$,

   (c) the sequence $(|f(T_n x)|)$ is bounded for every $x \in X$ and every $f \in Y^*$.

   Proof. We prove in the order (a) $\implies$ (b) $\implies$ (c) $\implies$ (a).

   (a) $\implies$ *(b)* There exists $M > 0$ such that $\sup_n \|T_n\| \leq M$. Fix any $x \in X$. Then for all $n \in \mathbb{N}$,

   $$\|T_n x\| \leq \|T_n\| \|x\| \leq M\|x\| < \infty.$$

   (b) $\implies$ *(c)* Fix any $x \in X$, there exists $M_x > 0$ such that $\sup_n \|T_n x\| \leq M_x$. Fix any $f \in Y^*$. Then for all $n \in \mathbb{N}$,

   $$\|f(T_n x)\| \leq \|f\| \|T_n x\| \leq \|f\| M_x < \infty.$$

   (c) $\implies$ *(a)* Let $Q : Y \to Y^{**}$ denote the natural embedding. Fix any $x \in X$. Since Y^* is a Banach space and for every $f \in Y^*$, by (c) we have

   $$\|Q(T_n x)(f)\| = \|f(T_n x)\| < \infty,$$

   it follows from Uniform Boundedness Theorem that there exists $M_x > 0$ (independent of f) such that for all $n \in \mathbb{N}$,

   $$\|T_n x\| = \|Q(T_n x)\| \leq M_x < \infty.$$

   Since X is a Banach space and the above inequality holds from all $x \in X$, by Uniform Boundedness Theorem there exists $M > 0$ such that $\|T_n\| \leq M$ for all $n \in \mathbb{N}$.

   □

2. Show that the space

   $$Y = \{X \in C^1[0, 1] : x(0) = 0\}$$

   equipped with the sup-norm is not a Banach space (cf. the following lemma).

   Lemma 1. *The sequence*

   $$x_n(t) = \sqrt{(t - \frac{1}{2})^2 + \frac{1}{n}}, \qquad t \in [0, 1]$$

   converges uniformly to the function $x(t) = |t - 1/2|$ *on* $[0, 1]$.

Proof. For $n \in \mathbb{N}$, define $y_n = x_n - \sqrt{\frac{1}{4} + \frac{1}{n}}$. Then $y_n(0) = 0$ and

$$y_n' = \frac{t - 1/2}{\sqrt{(t-1/2)^2 + 1/n}} \in C[0,1],$$

thus $y_n \in C^1[0,1]$ for all $n \in \mathbb{N}$.

It follows from Lemma 1 that $x_n \to x$ in sup-norm. Since $\sqrt{\frac{1}{4} + \frac{1}{n}} \to \frac{1}{2}$ in sup-norm as $n \to \infty$, we have $y_n = x_n - \sqrt{\frac{1}{4} + \frac{1}{n}} \to x - \frac{1}{2}$ as $n \to \infty$ in sup-norm. Hence $(y_n)_{n=1}^{\infty}$ is a Cauchy sequence in $C^1[0,1]$ with respect to sup-norm.

Suppose on the contrary that $(C^1[0,1], \text{sup-norm})$ is a Banach space. Then there exists $y^* \in C^1[0,1]$ such that $\lim_{n\to\infty} y_n = y^*$ in sup-norm. Then

$$y^*(t) = \lim_{n\to\infty} y_n(t) = x(t) - \frac{1}{2} \quad , \forall\, t \in [0,1].$$

Hence $y^* = x - \frac{1}{2}$ but not differentiable at $t = \frac{1}{2}$, thus $y^* \notin C^1[0,1]$ which is a contradiction. □

— *THE END* —