MATH3290 Mathematical Modeling 2021/22

Assignment #4

Due Date: 5pm, Apr. 24

Note For Problems 1, 3 and 4, give the steps clearly and hand in your assignment to the *assignment box* in LSB; for Problem 2, pack your codes and results into a zip-file **[your name]-[your student ID]-assign4.zip**, then email it to TA (**zqwang@math.cuhk.edu.hk**). "**(Optional)**" means this problem is optional and solving or not depends on your own will.

Problem 1

Consider the following economic model: Let P be the price of a single item on the market. Let Q be the quantity of the item available on the market. Both P and Q are functions of time. If we consider price and quantity as two interacting species, the following model might be proposed as follows

$$\begin{aligned}\frac{\mathrm{d}P}{\mathrm{d}t} &= a\left(\frac{b}{Q} - P\right)P,\\ \frac{\mathrm{d}Q}{\mathrm{d}t} &= c(fP - Q)Q,\end{aligned}$$

where a, b, c and f are *positive* constants.

(a) Find the equilibrium points of this system in terms of the constants a, b, c and f.

(b) If $a = 1$, $b = 20{,}000$, $c = 1$ and $f = 30$, calculate the equilibrium points of this system using the result of (a).

(c) Perform a graphical stability analysis to determine what will happen to the levels of P and Q as time increase. Also, classify each equibrium piont with respect to its stability, if possible. If a point cannot be readily classified, explain the reason.

(d) **(Optional)** Give an economic interpretation of the curves that determine the equilibrium points.

Problem 2

Consider launching a satellite into a orbit using a single-stage rocket. The rocket is continuously losing mass, which is being propelled away from it at significant speeds. We are interested in predicting the maximum speed the rocket can attain.

(a) Assume the rocket of mass m is moving with speed v. In a small increment of time Δt it loses a small mass Δm_p, which leaves the rocket with speed u in a direction opposite to v. Here, Δm_p is the small propellant mass. The resulting speed of the rocket is $v + \Delta v$. Neglect all external forces (gravity, atmospheric drag, etc.) and assume Newton's second law of motion:

$$\text{force} = \frac{\mathrm{d}}{\mathrm{d}t}(\text{momentum of system}).$$

where momentum is mass times velocity. Derive the model

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \left(\frac{-v_{\mathrm{e}}}{m}\right)\frac{\mathrm{d}m}{\mathrm{d}t},$$

where $v_e = u + v$ is the *relative* exhaust speed (the speed of the burnt gases relative to the rocket).

(b) Assume that initially, at time $t = 0$, the velocity $v = 0$ and the mass of the rocket is $m = M + P$, where P is the mass of the payload satellite and $M = \varepsilon M + (1 - \varepsilon)M$ $(0 < \varepsilon < 1)$ is the initial fuel mass εM plus the mass $(1 - \varepsilon)M$ of the rocket casings and instruments. Solve the model in (a) to obtain the speed

$$v(t) = -v_e \ln\left[\frac{m(t)}{M + P}\right].$$

(c) Show that when all fuel is burned, the speed of the rocket is given by

$$v_f = -v_e \ln\left[1 - \frac{\varepsilon}{1 + \beta}\right]$$

where $\beta = P/M$ is the ratio of the payload mass to the rocket mass.

(d) Find v_f if $v_e = 3\text{km/sec}$, $\varepsilon = 0.8$ and $\beta = 0.01$. (These are typical values in satellite launchings.)

Problem 3

(Controlling a population) The fish and game department in a certain state is planning to issue hunting permits to control the deer population (one deer per permit). It is known that if the deer population falls below a certain level m, the deer will become extinct. It is also known that if the deer population rises above the carrying capacity M, the population will decrease back to M through disease and malnutrition. Assume that P is the population of the deer and r is a positive constant of proportionality. The model can be formulated as follows

$$\frac{\mathrm{d}P}{\mathrm{d}t} = rP(M - P)(P - m),$$

where $0 < m < M$ and $0 < r$.

(a) Write down the explicit formula for the population P in terms of r, m, M and $P(0)$. (hint: use the identity

$$\frac{1}{z(z-M)(z-m)} = \frac{1}{mMz} - \frac{1}{m(M-m)(z-m)} + \frac{1}{M(M-m)(z-M)}$$

to help to find the explicit formula.)

(b) Show that if $P > M$ for all t, then we have

$$\lim_{t\to\infty} P(t) = M.$$

(c) What happens if $P < M$ for all t?

(d) What are the equilibrium points of the model? Explain the dependence of the steady-state value of P on the initial values of $P(0)$.

Problem 4

(Programming exercise) In this exercise, you are required to implement the Euler's method and do the parameter identification. Consider the following model of differential equation defined on the time interval $[0, T]$,

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} &= af(t, y) + bg(t, y), \\ y(0) &= \alpha, \end{aligned}$$

where $T = 2$ and the model functions $f(t, y)$ and $g(t, y)$ are given as follows:

$$f(t, y) = y \quad \text{and} \quad g(t, y) = t(1 + \sin y).$$

You are asked to determine the parameters a and b for the model with the set of initial conditions α's and responses β's at time $t = T$ as follows:

α	0	0.6	0.9	1.4	1.7
β	2.0	2.4	1.8	1.6	1.5

Complete the main code in the file a4q2.m and plot the parameters a_k and b_k against k. Here are the step-by-step instructions for this exercise:

(a) Write a **MATLAB code** to implement the Euler's method. You should write your commands in the m-file euler.m and follow the hints in the same m-file to complete your code. Set the time step $\Delta t = 0.02$. Assign the initial guess: $a_0 = 1$, $b_0 = 0.5$ and set $k = 0$.

(b) Find $y_i(t; a, b)$ $(i = 1, \cdots, 5)$ by solving the following ODE over $[0, 2]$ via Euler's method

$$\begin{aligned} \frac{\mathrm{d}y_i}{\mathrm{d}t} &= a_k f(t, y_i) + b_k g(t, y_i), \\ y_i(0) &= \alpha_i. \end{aligned}$$

After solving the ODE, one should obtain the value of $y_i^n = y_i(t_n)$ at each point $t_n = n\Delta t$ where $n = 0, \cdots, 100$.

(c) Estimate $A_i(T; a_k, b_k) = \frac{\partial y_i}{\partial a}(T; a, b)$ by solving the following ODE using Euler's method (use the **MATLAB code** written in (a) to solve)

$$\begin{aligned} \frac{\mathrm{d}A_i}{\mathrm{d}t} &= f(t, y_i) + \big(a_k f_y(t, y_i) + b_k g_y(t, y_i)\big) A_i, \\ A_i(0) &= 0. \end{aligned}$$

Note that, we can use the value of y_i^n obtained from (b) in the computation.

(d) Similarly, estimate $B_i(T; a_k, b_k) = \frac{\partial y_i}{\partial b}(T; a, b)$ by solving the following ODE using Euler's method

$$\begin{aligned} \frac{\mathrm{d}B_i}{\mathrm{d}t} &= g(t, y_i) + \big(a_k f_y(t, y_i) + b_k g_y(t, y_i)\big) B_i, \\ B_i(0) &= 0. \end{aligned}$$

(e) Set $\lambda_k = 0.005$ and update a_{k+1} and b_{k+1} by the following formula

$$\begin{aligned} a_{k+1} &= a_k - \lambda_k \frac{\partial S}{\partial a}(a_k, b_k), \\ b_{k+1} &= b_k - \lambda_k \frac{\partial S}{\partial b}(a_k, b_k), \end{aligned}$$

where

$$\begin{aligned} \frac{\partial S}{\partial a}(a_k, b_k) &= -2\lambda_k \sum_{i=1}^{5} (\beta_i - y_i(T; a_k, b_k))\, A_i(T; a_k, b_k), \\ \frac{\partial S}{\partial b}(a_k, b_k) &= -2\lambda_k \sum_{i=1}^{5} (\beta_i - y_i(T; a_k, b_k))\, B_i(T; a_k, b_k). \end{aligned}$$

(f) If we have

$$\sqrt{\left(\frac{\partial S}{\partial a}\right)^2 + \left(\frac{\partial S}{\partial b}\right)^2} < 10^{-6},$$

or $k > 100$, then we stop. Otherwise, set $k \leftarrow k + 1$ and repeat the calculation from (b) to (e).