Problem 2

(a) $C_1^{n+1} = C_1^n 0.3 + C_2^n 0.3 + C_3^n 0.4$

$C_2^{n+1} = C_1^n 0.2 + C_2^n 0.5 + C_3^n 0.2$

$C_3^{n+1} = C_1^n 0.5 + C_2^n 0.2 + C_3^n 0.4$

(b) $C^n = [C_1^{n+1}, C_2^{n+1}, C_3^{n+1}]^T$ $\quad C^{n+1} = TC^n$ $\quad T = \begin{pmatrix} 0.3 & 0.3 & 0.4 \\ 0.2 & 0.5 & 0.2 \\ 0.5 & 0.2 & 0.4 \end{pmatrix}$

$C^* = TC^* \Rightarrow (E - T)C^* = 0$ $\qquad E = \begin{pmatrix} 1.0 & 0 & 0 \\ 0 & 1.0 & 0 \\ 0 & 0 & 1.0 \end{pmatrix}$

$$\begin{pmatrix} 0.7 & -0.3 & -0.4 \\ -0.2 & 0.5 & -0.2 \\ -0.5 & -0.2 & 0.6 \end{pmatrix} \to \begin{pmatrix} 1 & -\frac{3}{7} & \frac{-4}{7} \\ 0 & \frac{29}{7} & \frac{-22}{7} \\ 0 & -\frac{29}{7} & \frac{22}{7} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -\frac{26}{29} \\ 0 & 1 & -\frac{22}{29} \\ 0 & 0 & 0 \end{pmatrix}$$

$C^* = [C_1^*, C_2^*, C_3^*]^T \Rightarrow \begin{cases} C_1^* - \frac{26}{29}C_3^* = 0 \\ C_2^* - \frac{22}{29}C_3^* = 0 \end{cases}$ $\underset{\sim}{+}$ $C_1^* + C_2^* + C_3^* = 1$

$$\Rightarrow \begin{cases} C_1^* = \frac{26}{77} \\ C_2^* = \frac{22}{77} = \frac{2}{7} \\ C_3^* = \frac{29}{77} \end{cases}$$

<u>Suppose</u> T has 3 <u>different</u> eigenvalues, that are $\lambda_1, \lambda_2, \lambda_3$

with corresponding eigenvectors v_1, v_2, v_3. $(Tv_i = \lambda_i v_2 \quad i = 1, 2, 3)$

<u>Then</u> if $C^0 = k_1 v_1 + k_2 v_2 + k_3 v_3$

$$C^n = k_1 \lambda_1^n v_1 + k_2 \lambda_2^n v_2 + k_3 \lambda_3^n v_3$$

Let's Calculate the eigenvalues of T ($\lambda_1 = 1$ from $\det(E-T) \neq 0$!)

$$\det(\lambda E - T) = \begin{vmatrix} \lambda-0.3 & -0.3 & -0.4 \\ -0.2 & \lambda-0.5 & -0.2 \\ -0.5 & -0.2 & \lambda-0.4 \end{vmatrix} = (\lambda-0.3)\begin{vmatrix} \lambda-0.5 & -0.2 \\ -0.2 & \lambda-0.4 \end{vmatrix} + 0.2\begin{vmatrix} -0.3 & -0.4 \\ -0.2 & \lambda-0.4 \end{vmatrix} \pm 0.5\begin{vmatrix} -0.3 & -0.4 \\ \lambda-0.5 & -0.2 \end{vmatrix}$$

$$= (\lambda-0.3)\Big((\lambda-0.5)(\lambda-0.4)-0.04\Big)+0.2\Big(-0.3(\lambda-0.4)-0.08\Big)-0.5\Big(0.06+0.4(\lambda-0.5)\Big)$$

$$= \lambda^3-1.2\lambda^2+0.17\lambda+0.03$$

$$= (\lambda-1)\lambda^2-0.2\lambda^2+0.17\lambda+0.03 = (\lambda-1)\lambda^2-0.17\lambda(\lambda-1)-0.03(\lambda-1)(\lambda+1)$$

$$= (\lambda-1)\Big(\lambda^2-0.2\lambda-0.03\Big) = (\lambda-1)(\lambda-0.3)(\lambda+0.1)$$

$\lambda_1=1 \quad \lambda_2=0.3 \quad \lambda_3=-0.1 \qquad C^n = k_1\lambda_1^n v_1 + k_2\lambda_2^n v_2 + k_3\lambda_3^n v_3$

$k_2\lambda_2^n \xrightarrow[n]{} 0 \quad k_3\lambda_3^n \xrightarrow[n]{} 0 \qquad C^n \xrightarrow[n]{} k_1 v_1$

$T v_2 = \lambda_2 v_2 \qquad e = [1, 1, 1] \qquad T^T e = e \;! \qquad C^0 \cdot e = 1 = C_1^0 + C_2^0 + C_3^0$

$$e\cdot T v_2 = v_2\cdot T^T e = v_2\cdot e \overset{(\lambda_2\neq 1)}{=} \lambda_2 v_2\cdot e \Rightarrow v_2\cdot e = 0\;!$$

Similarly, $v_3\cdot e = 0$!

Choose $v_1 = C^*$ $\quad 1 = C^0\cdot e = (k_1 v_1 + k_2 v_2 + k_3 v_3)\cdot e$

$$= k_1\underbrace{(v_1\cdot e)}_{(=1)} + k_2\underbrace{(v_2\cdot e)}_{(=0)} + k_3\underbrace{(v_3\cdot e)}_{(=0)} = k_1$$

$\Rightarrow \boxed{C^n \xrightarrow[n]{} v_1}$! $\quad \boxed{C^n \xrightarrow[n]{} C^*}$

Problem 3

l	14.5	12.5	17.25	14.5	12.625	17.75	14.125	12.625
g	9.75	8.375	11.0	9.75	8.5	12.5	9.0	8.5
W	27	17	41	26	17	49	23	16
g^3								
gl^2								
g^6								
g^3W			Fill in data					
g^2l^4								
g^2l^2W								

$$S_1(c) = \sum_i (W_i - c g_i^3)^2 \qquad S_2(k) = \sum_i (W_i - k g_i l_i^2)^2$$

$$\frac{\partial S_1}{\partial c} = \sum_i -g_i^3 (W_i - c g_i^3) = c\left(\sum_i g_i^6\right) - \sum_i g_i^3 W_i$$

$$c^* = \frac{\sum_i g_i^3 W_i}{\sum_i g_i^6} \approx 0.0276 \qquad S_1(c^*) \approx 54.2563$$

$$\frac{\partial S_2}{\partial k} = \sum_i -g_i l_i^2 (W_i - k g_i l_i^2) = k\left(\sum_i g_i^2 l_i^4\right) - \sum_i g_i l_i^2 W_i$$

$$k^* = \frac{\sum_i g_i l_i^2 W_i}{\sum_i g_i^2 l_i^4} \approx 0.0126 \qquad S_2(k^*) \approx 3.3885$$

$S_1(c^*) \gg S_2(k^*)$ $\quad$ ($W \propto g l^2$) is better!