MATH3290 Mathematical Modeling 2021/22

Supplementary Materials

January 28, 2022

Consider a (multidimensional) linear dynamic system

$$x_{n+1} = Ax_n + b, \tag{1}$$

where $x_n, b \in \mathbb{R}^d$ and $A \in \mathbb{R}^{n\times n}$, we assume x^* is an equilibrium point of this dynamic system, which means $x_n \to x^*$ as $n \to \infty$. Take limits on both sides of eq. (1), we have

$$x^* = Ax^* + b, \tag{2}$$

which gives a *necessary* condition of x^* be an equilibrium point.

We first to determine eq. (2) is solvable or not (if we cannot find a solution, there must be no equilibrium points for the original dynamic system). If $\det(I - A) \neq 0$, then we can obtain an *unique* solution $(I - A)^{-1}b$ of eq. (2). However, this solution $(I - A)^{-1}b$ may not be an equilibrium point (e.g., $A = 2I$). I presented a *sufficient* condition in the class as $\sigma_{\max}(A) < 1$, which does not mean that $(I - A)^{-1}b$ will not be an equilibrium point if $\sigma_{\max}(A) \geq 1$. Actually, a better condition is the spectral radius[1] of A is less than 1, while it may need to work on the complex field $\mathbb{C}$ to make this theory self-consistent (the reason I skip introducing it).

Back to the assignment, we have $b = 0$ and $\det(I - A) = 0$. The equilibrium point must be one solution of $(I - A)x = 0$. A good message is that A seems to have n *distinctive* real eigenvalues (things will be tricky if there are complex eigenvalues) and we can also find $v_i \in \mathbb{R}^d$ such that $Av_i = \lambda_i v_i$. Eigenvectors $\{v_i\}$ are linearly independent, and we can hence assume

$$x_0 = \sum_{i=1}^{n} w_i v_i,$$

which leads to

$$x_n = \sum_{i=1}^{n} w_i \lambda_i^n v_i.$$

Now we can draw a conclusion about limiting points of this dynamic system. Generally, this depends on w_i of x_0 and λ_i of A. If there exists $1 \leq i \leq n$ with $w_i = 0$ and $\lambda_i \notin (-1, 1]$, then x_n *cannot* converge to x^*. To see this, it is possible to find a vector v such that $v \cdot v_{i'} = 0$ for $i' \neq i$ and $v \cdot v_i \neq 0$; then if $x_n \to x^*$, we have $x_n \cdot v = \lambda_i^n (w_i v_i \cdot v) \to x^* \cdot v$, which leads a contradiction since λ_i^n cannot converge w. r. t. n and $w_i v_i \cdot v \neq 0$.

We have the *first* argument: if there exists $1 \leq i \leq n$ with $\lambda_i \notin (-1, 1]$, this dynamic system cannot have equilibrium points (by setting a specific x_0, x_n will not converge).

Similarly, the *second* argument states as follows: if for any i we have $|\lambda_i| < 1$, then the dynamic system has an unique equilibrium point 0 (for any x_0, x_n will converge to 0).

Moreover, we have the *third* argument: if there exists $\lambda_i = 1$ (we assume eigenvalues are all distinctive) and the rest eigenvalues have the relation $|\lambda_{i'}| < 1$, then $x_n \to w_i v_i$where $x_0 = \sum_{i=1}^n w_i v_i$. We still call $w_i v_i$ as an equilibrium point , although it depends on the initial value x_0, because in our assignment, we have a hidden relation $\sum_{i=1}^n x^0[i] = 1$. By considering this restriction, we can obtain the *fourth* argument: for any x^0 satisfying $\sum_{i=1}^n x^0[i] = 1$, it holds that $x^n \to x^* = w_i v_i$. (To rigorously prove such a conclusion, you may take some time to solve **(Optional)** first).

A take-home message is the eigenvalue distribution of A determines the equilibrium points.

[1] You may check the Wiki page "Spectral radius"