MATH 3060 Tutorial 6

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1 Questions of this tutorials

1. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be injective and smooth, then f' is everywhere invertible.

   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth surjective function with f' everywhere invertible, then f has a smooth inverse.

   (c) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth surjective function with Df everywhere invertible, then f has a smooth inverse.

   (d) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable function with Df invertible at the origin, then f has a differentiable local inverse at the origin.

   (e) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth function with Df everywhere invertible, then f sends open sets to open sets.

   (f) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth function with Df everywhere invertible, then f sends closed sets to closed sets.

2. (Morse Lemma)Let $p \in \mathbb{R}^n$ be the origin, and $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function with $f(p) = p$ and $Df(p) = 0$. Let A be the matrix $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\Big|_p \right)_{i,j}$.

   (a) Let $M : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation (i.e. M is an $n \times n$ matrix), and let $g = f \circ M$. Show that

   $$\left(\frac{\partial^2 g}{\partial x_i \partial x_j}\Big|_p \right)_{i,j} = M^T A M$$

   (b) If A is positive definite, show that there exist open neighbourhoods U, V of p, a smooth map $\Phi : U \to V$ such that $D\Phi(p)$ is invertible, and

   $$f(\Phi(x_1, x_2, \ldots, x_n)) = x_1^2 + x_2^2 + \cdots + x_n^2.$$

   (c) What can be said if A has signature $(k, n-k)$?