MATH 3060 Assignment 4 solution

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1. (a) This is because for any $x \in X$ with $x \neq x_0$, we have $d(x, x_0) = 1 > \epsilon$.

   (b) For each $x \in X$, $\{x\} = B_{\frac{1}{2}}(x)$ is open, but every set is the union of its one point subset, so every subsets of X is open. Since a subset of X is closed if and only if its complement is open, every subset of X is closed as well.

2. (a) Yes. Let $g \in B^1_\epsilon(f)$, since $B^1_\epsilon(f)$ is open with respect to d_1, we can find an $r > 0$ such that $B^1_r(g) \subset B^1_\epsilon(f)$. But since $d_1 \leq d_\infty$, we have $B^\infty_r(g) \subset B^1_r(g)$.

   (b) No. Consider the function

$$f_n(x) = \begin{cases} 1 - nx & \textit{if } \epsilon(x \leq \frac{1}{n}) \\ 0 & \textit{if } x \geq \frac{1}{n} \end{cases}.$$

   Then $d_1(f_n, 0) \to 0$, this means that for any $r > 0$, $f_n \in B^1_r(0)$ for n large enough. On the other hand, $d_\infty(f_n, 0) = \epsilon$, which means that $f_n \notin B^\infty_\epsilon(0)$ for any n. Therefore, $B^\infty_\epsilon(0)$ cannot be open with respect to d_2.

3. (a) It is clear that $d(x, y) = d(y, x)$, $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$. Moreover, for $x, y, z \in l_2$. We want to show

$$\begin{aligned}
& d(x,y) + d(y,z) \geq d(x,z) \\
\iff & \sqrt{\sum (x_i - y_i)^2} + \sqrt{\sum (y_i - z_i)^2} \geq \sqrt{\sum (x_i - z_i)^2} \\
\iff & \left(\sqrt{\sum (x_i - y_i)^2} + \sqrt{\sum (y_i - z_i)^2} \right)^2 \geq \sqrt{\sum [(x_i - y_i) + (y_i - z_i))]^2} \\
\iff & \sqrt{\sum (x_i - y_i)^2 \sum (y_i - z_i)^2} \geq \sum (x_i - y_i)(y_i - z_i) \\
\iff & \sum (x_i - y_i)^2 \sum (y_i - z_i)^2 \geq \sum (x_i - y_i)^2 (y_i - z_i)^2
\end{aligned}$$

   which is the Cauchy Schwartz inequality.

(b) Suppose $x^n \in H$ with $\lim x^n = x$. We need to show that $x_i \leq 1/i$ for each i. But in fact

$$|x_i - x_i^n|^2 < \sum_{j=1}^{\infty} |x_j - x_j^n|^2 \to 0.$$

We see that $x_i = \lim x_i^n \leq 1/i$, so $x \in H$. This shows that H is closed.

4. (a) It is open but not closed. It is open because for $x \in [a, c)$, we have $B_r(x) \subset [a, c)$ for $r = \frac{1}{2}(c - x)$. It is not closed because $\lim_n (c - 1/n) = c \notin [a, c)$.

   (b) It is open but not closed. It is open because for $x \in (c, b)$, we have $B_r(x) \subset [a, c)$ for $r = \frac{1}{2} \min\{x - c, b - x\}$. It is not closed because $\lim_n (c + 1/n) = c \notin (c, b)$.

   (c) The only subsets of $[a, b)$ which are both open and closed are $[a, b)$ and $\emptyset$. In fact, suppose $U \subset [a, b)$ is both open and closed.
   Suppose $U \neq [a, b), \emptyset$. Since U nonempty, $\inf U$ exists and $\geq a$. We claim that $\inf U$ must be a. If $\inf U = c > a$, then because U is closed, we must have $c \in U$. But then since U is open, $x - \epsilon \in U$ for some small ϵ, this contradicts to $c = \inf U$. We thus have $a = \inf U$, and hence $a \in U$.
   However, we can apply the same argument to the complement $V = [a, b) \backslash U$ of U to conclude $a \in V$. This is a contradiction because we cannot have $a \in U \cap V = \emptyset$.