§4.2 Baire Category Theorem

Def Let (X,d) be a metric space. A set E in X is <u>dense</u> if $\forall\, x \in X$ and $\varepsilon > 0$,

$$B_\varepsilon(x) \cap E \neq \phi$$

Notes: (i) Easy to see that E is dense $\Leftrightarrow$ $\overline{E} = X$.
(ii) X is dense (in (X,d))

eg: If $(X$, discrete metric$)$, then for $0<\varepsilon<1$ and $x \in X$, $B_\varepsilon(x) = \{x\}$. Therefore E is dense in X $\Rightarrow$ $E = X$ (i.e. X is the only dense set in $(X$, discrete$)$)

eg1: In $(\mathbb{R}$, standard metric$)$, $\mathbb{Q}$ and $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$ are dense.

eg2: Weierstrass approximation theorem implies the set of all polynomials $\mathcal{P}$ forms a dense set in $(C[0,1], d_\infty)$.

Def: Let (X,d) be a metric space. A subset $E \subset X$ is called <u>nowhere dense</u> if its <u>closure</u> does not contain any metric ball.
(i.e. $\overline{E}$ has empty interior $(\overline{E})^\circ = \phi$)

eg. • $\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$ nowhere dense in $\mathbb{R}$.
• $\mathbb{Q}$ has empty interior, but $\overline{\mathbb{Q}} = \mathbb{R}$ has nonempty interior, so $\mathbb{Q}$ is <u>not</u> nowhere dense.