Unless otherwise specified, if we write $(a,b)$ or $[a,b]$, it is always the case that $a < b \in \mathbb{R}$.

# 1 Norms on Vector Spaces

**Definition 1.1.** Let $X$ be a vector space. We say a function $\|\cdot\| : X \to \mathbb{R}$ is a norm if

i. $\|x\| \geq 0$ for all $x \in X$ (non-negativity)

ii. $\|\alpha x\| = |\alpha|\|x\|$ for all $x \in X$ (scalar homogeneity)

iii. $\|x+y\| \leq \|x\| + \|y\|$ (triangle inequality)

iv. $\|x\| = 0$ if and only if $x = 0$ (separating points)

If there is a norm $\|\cdot\|$ on $X$, we call the pair $(X, \|\cdot\|)$ a normed space.

## Quick Practice

1. Show that the absolute value function $|\cdot| : \mathbb{R} \to \mathbb{R}$ is a norm on the vector space of $\mathbb{R}$.
   **Solution.** Covered in Tutorial.

2. Let $\mathcal{C}([0,1])$ denote the space of continuous functions on $[0,1]$.
   a. Show that $\mathcal{C}([0,1])$ is a vector space.
   b. Define for all $f \in \mathcal{C}([0,1])$ that $\|f\|_\infty := \sup_{t\in[0,1]} |f(t)|$. Show that $\|\cdot\|_\infty$ is a norm of $\mathcal{C}([0,1])$.

   **Solution.** Covered in Tutorial.

3. Let $\ell_\infty$ denote the space of bounded real-valued sequence, that is, $x \in \ell_\infty$ if and only if $x = (x_n)$ where $(x_n)$ is a bounded sequence and $x_n \in \mathbb{R}$.
   a. Show that $\ell_\infty$ is a vector space (with the obvious operations).
   b. Define for all $x = (x_n) \in \ell_\infty$ that $\|x\|_\infty := \sup_n |x_n|$. Show that $\|\cdot\|_\infty$ is a norm on $\ell_\infty$.
   c. Denote $c_0$ the space of sequences that converge to 0, that is, $x := (x_n) \in c_0$ if and only if $\lim x_n = 0$. Show that $c_0 \subset \ell_\infty$ and it is a vector subspace of $\ell_\infty$.
   d. Show that $\|\cdot\|_\infty$ is a norm on $c_0$.

   **Solution.** Covered in Tutorial. All are not hard, except maybe for the notations.

4. Let $\text{Lip}([0,1])$ denote the space of Lipschitz functions on $[0,1]$. Define $\|f\|_L := \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|}$.
   a. Show that $\text{Lip}_0([0,1])$ is a vector space and $\|\cdot\|_L$ satisfies $(i) - (iii)$ in the definition of a norm.
   b. Show that $\|\cdot\|_L$ is NOT a norm.
   c. Define $\text{Lip}_0([0,1]) := \{f \in \text{Lip}([0,1]) : f(0) = 0\}$. Show that $\text{Lip}_0([0,1])$ is a vector subspace of $\text{Lip}([0,1])$ and that $\|\cdot\|_L$ is a norm on $\text{Lip}_0([0,1])$.

   **Solution.** Covered in Tutorial.

5. Consider the space of Riemann integrable functions $\mathcal{R}([0,1])$. Define $\|f\|_1 := \int_0^1 |f|$ for all $f \in \mathcal{R}([0,1])$.
   a. Show that $(\mathcal{C}([0,1]), \|\cdot\|_1)$ is a normed space.
   b. Is $\|\cdot\|_1$ a norm on $\mathcal{R}([0,1])$ ?

   **Solution.** (a). The trickiest part is how $\|f\|_1 = 0$ would imply $f = 0$ for continuous $f$. This follows from HW5 Q2. (b). No. It is not a norm as $\|f\|_1 = 0$ would not imply $f = 0$ for general integrable functions.

## More Quick Practice

1. Let $X$ be a normed space. We say that a subset $C \subset X$ is convex if for all $x, y \in C$ we have $tx+(1-t)y \in C$ for all $t \in [0,1]$. Define $B(x,r) := \{y \in X : \|y-x\| < r\}$ for all $x \in X$ and $r > 0$.

   a. Show that $B(0,1)$ is convex.

   b. Show that $B(x,r)$ is convex for all $x \in X$ and $r > 0$

   **Solution.** (a), (b): Let $a, b \in B(x,r)$ and $t \in [0,1]$. Then $ta+(1-t)b-x = t(a-x)+(1-t)(b-x)$. Hence, we have

$$\|t(a-x)+(1-t)(b-x)\| \leq t\|a-x\| + (1-t)\|b-x\| < tr + (1-t)r = r$$

2. Let $p \geq 1$. Define for all $f \in \mathcal{C}([0,1])$ that $\|f\|_p := (\int_0^1 |f|^p)^{1/p}$.

   a. Show that $\|\cdot\|_p$ satisfies (i), (ii), (iv) in the definition of a norm for all $p \geq 1$.

   b. When $p = 1, 2$, show that $\|\cdot\|_p$ is a norm.

   c. (Challenging, cf Tutorial 3). Show that $\|\cdot\|_p$ is a norm for all $p \geq 1$.

   **Solution.** (a). Easy. (b). $p = 1$ follows from the triangle inequality. $p = 2$ follows from the Cauchy-Schwarz inequality (cf. HW6 Q3) with techniques from Tutorial 3). (c). Not much different from the solutions in Tutorial 3, P.2 Q3-5.

3. Let $X$ be a normed space. Let $(x_n)$ be a sequence in $X$. We say that

   - $(x_n)$ is a Cauchy sequence in $X$ if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|x_n - x_m\| < \epsilon$.
   - $(x_n)$ converges to $x \in X$ if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|x_n - x\| < \epsilon$. In other words, $\lim_n \|x_n - x\| = 0$.

   Now consider $X = (\mathcal{C}([0,1]), \|\cdot\|_\infty)$. Let $(f_n)$ be a Cauchy sequence in $X$.

   a. Show that for all $t \in [0,1]$, the sequence $(f_n(t))$ is a Cauchy sequence in $\mathbb{R}$.

   b. Following (a), define $f(t) := \lim_n f_n(t)$ for all $t \in [0,1]$. Show that $(f_n)$ converges to $f$.

   c. (Tricky). Consider now instead $Y := (\mathcal{C}([0,1]), \|\cdot\|_1)$, where $\|f\|_1 := \int_0^1 |f|$. Show that not every Cauchy sequence in $Y$ converges.
   *(Hint: Find a sequence $(f_n)$ in $\mathcal{C}([0,1])$ and $f \in \mathcal{R}([0,1]) \backslash \mathcal{C}([0,1])$ such that* $\lim \|f_n - f\|_1 = 0$.

   **Solution.** (a), (b) are easy facts about uniform convergence. (c). Consider $f : [0,1] \to \mathbb{R}$ with $f(x) = 1$ for all $x \in [0, \frac{1}{2})$ and $f(x) = 0$ for $x \in [\frac{1}{2}, 1]$. Let $(r_n)$ be a sequence of positive numbers $< 1/2$ such that $r_n \to 0$. Define for all $n \in \mathbb{N}$ that $f_n(x) = f(x)$ except for $x \in (\frac{1}{2} - r_n, \frac{1}{2})$ while $f_n$ is the line connecting $(\frac{1}{2} - r_n, 1)$ and $(\frac{1}{2}, 0)$, that is, $f_n(x) = \frac{-1}{r_n}(x - \frac{1}{2})$ for $x \in (\frac{1}{2} - r_n, \frac{1}{2})$. It is clear that $(f_n)$ is a sequence of continuous functions. It is not hard to see that we have for all $n \in \mathbb{R}$ that

$$\int_0^1 |f_n - f| = \int_{1/2-r_n}^{1/2} |f_n - f| = \int_{1/2-r_n}^{1/2} \frac{-1}{r_n}(x - \frac{1}{2})dx = \frac{-1}{r_n}[\frac{1}{2}x^2 - \frac{1}{2}x]_{1/2-r_n}^{1/2} = r_n$$

   Hence, we have $\|f_n - f\|_1 \to 0$ as $n \to \infty$ by Squeeze Theorem.
   Note that the fact that $\|f_n - f\|_1 \to 0$ would imply $(f_n)$ to be a Cauchy sequence in $\|\cdot\|_1$. Nonetheless it does not converge to a continuous function. Suppose not. Then $g$ is a continuous function with $\|f_n - g\|_1 \to 0$. This would imply $\int_0^1 |f - g| = 0$ and so $\int_0^{1/2} |f - g| = 0$ and $\int_{1/2}^1 |f - g| = 0$. Hence $g = 1$ on $[0, 1/2]$ while $g = 0$ on $[1/2, 1]$, which cannot be continuosu on $[0, 1]$.

4. Let $X$ be a normed space. We say that $f : X \to \mathbb{R}$ is continuous at $x \in X$ if for all $\epsilon > 0$, there exists $\delta > 0$ such that $\|x - y\| < \delta$ would imply $|f(x) - f(y)| < \epsilon$.

   a. Show that a linear functional $f : X \to \mathbb{R}$ is continuous for all $x \in X$ if and only if $f$ is continuous at $0 \in X$.

   b. Suppose $X = (\mathbb{R}, |\cdot|)$. Show that every linear functional is continuous.

   **Solution.** (a). Note that $f(x-y) = f(x) - f(y)$ for all $x, y \in X$ by linearity. The result follows clearly. (b). Note that $f(x) = f(x \cdot 1) = f(1)x$ for all $x \in \mathbb{R}$ and linear functional $f$. It is clearly that $f$ is continuous.

5. This question follows Q4. Let $X$ be a normed space. Then we denote $X^*$ the space of *continuous* linear functionals, that is, $f \in X^*$ if and only if $f$ is continuous and linear.

   a. Show that a linear functional $f \in X^*$ if and only if there exists $M > 0$ such that $|f(x)| \leq M\|x\|$

   b. Define for all $f \in X^*$ that $\|f\|_{X^*} := \sup_{\|x\| \leq 1} |f(x)|$. Show that $(X^*, \|\cdot\|_{X^*})$ is a normed space.

   c. (Challenging). Show that $(X^*, \|\cdot\|_{X^*})$ satisfies the Cauchy criteria, that is, every Cauchy sequence in $X^*$ converges to some element in $X^*$.

   **Solution.** (a). ($\Leftarrow$). This implies that $f$ is continuous at 0 and so $f$ is continuous by Q4. ($\Rightarrow$). Note that by continuity at 0 there exists $\delta > 0$ such that $|f(x)| \leq 1$ if $\|x\| < \delta$. Now pick $x \in X$. We consider $x_0 := \frac{\delta}{2} \frac{x}{\|x\|}$. Then $\|x_0\| < \delta$. It follows that $|f(x_0)| \leq 1$. This implies that $|f(x)| \leq \frac{2}{\delta}\|x\|$ by linearly for all $x \in X$.
   (b). is not hard and standard. (c). The argument is the very similar to why uniformly Cauchy sequence of continuous functions (from $A \subset \mathbb{R}$ to $\mathbb{R}$) has a continuous limit. Construct first a point-wise limit of $(f_n)$ using the Cauchy criteria on real numbers. Then show that the point-wise limit is a uniform limit by triangle inequalities.

6. Let $X, Y$ be a normed space. We say that $T : X \to Y$ is an isometry if $\|Tx\|_Y = \|x\|_X$ for all $x \in X$.

   a. Show that every isometry is injective.

   b. For $f \in \mathcal{C}^1([0,1])$, define $\|f\|_L := \sup_{x \neq y} \frac{|f(x)-f(y)|}{|x-y|}$ and $\|f\|_S := \|f'\|_\infty = \sup_{t \in [0,1]} |f'(t)|$. Show that $\|\cdot\|_L$ and $\|\cdot\|_S$ satisfies (i) to (iii) in the definition of a norm but is not a norm on $\mathcal{C}^1([0,1])$

   c. Consider the subspace $Z := \mathcal{C}_0^1([0,1]) := \{f \in \mathcal{C}(0,1) : f(0) = 0\}$. Show that $(Z, \|\cdot\|_L)$ and $(Z, \|\cdot\|_S)$ are normed spaces.

   d. Consider $X := (Z, \|\cdot\|_L)$ and $Y := (Z, \|\cdot\|_S)$. Show that there exists a *surjective linear* isometry, or a linear isometric isomorphism, between $X$ and $Y$.

   **Solution.** (a) to (c) are standard and not hard; the proofs are omitted. (d). Just consider the identity map of $Z$, which becomes a linear map between $X, Y$.

7. (Functional Interpretation of FTC, cf. Tutorial 7). Let $X := (\mathcal{C}([0,1]), \|\cdot\|_\infty)$ and $Y := (\mathcal{C}_0^1([0,1]), \|\cdot\|_L)$, which is defined in Q6. Define $T : X \to Y$ by $Tf(t) := \int_0^t f$ for all $t \in [0,1]$. Show that $T$ is a linear isometric isomorphism between $X, Y$ by finding explicitly the inverse of $T$.

   **Solution.** Covered in Tutorial 7. Here is a restatement only.