Unless otherwise specified, if we write (a, b) or $[a, b]$, it is always the case that $a < b \in \mathbb{R}$.

1 Norms on Vector Spaces

Definition 1.1. Let X be a vector space. We say a function $\|\cdot\| : X \to \mathbb{R}$ is a norm if

i. $\|x\| \geq 0$ for all $x \in X$ (non-negativity)

ii. $\|\alpha x\| = |\alpha|\|x\|$ for all $x \in X$ (scalar homogeneity)

iii. $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality)

iv. $\|x\| = 0$ if and only if $x = 0$ (separating points)

If there is a norm $\|\cdot\|$ on X, we call the pair $(X, \|\cdot\|)$ a normed space.

Quick Practice

1. Show that the absolute value function $|\cdot| : \mathbb{R} \to \mathbb{R}$ is a norm on the vector space of $\mathbb{R}$.

2. Let $\mathcal{C}([0, 1])$ denote the space of continuous functions on $[0, 1]$.

   a. Show that $\mathcal{C}([0, 1])$ is a vector space.

   b. Define for all $f \in \mathcal{C}([0, 1])$ that $\|f\|_\infty := \sup_{t\in[0,1]} |f(t)|$. Show that $\|\cdot\|_\infty$ is a norm of $\mathcal{C}([0, 1])$.

3. Let ℓ_∞ denote the space of bounded real-valued sequence, that is, $x \in \ell_\infty$ if and only if $x = (x_n)$ where (x_n) is a bounded sequence and $x_n \in \mathbb{R}$.

   a. Show that ℓ_∞ is a vector space (with the obvious operations).

   b. Define for all $x = (x_n) \in \ell_\infty$ that $\|x\|_\infty := \sup_n |x_n|$. Show that $\|\cdot\|_\infty$ is a norm on ℓ_∞.

   c. Denote c_0 the space of sequences that converge to 0, that is, $x := (x_n) \in c_0$ if and only if $\lim x_n = 0$. Show that $c_0 \subset \ell_\infty$ and it is a vector subspace of ℓ_∞.

   d. Show that $\|\cdot\|_\infty$ is a norm on c_0.

4. Let $\mathrm{Lip}([0, 1])$ denote the space of Lipschitz functions on $[0, 1]$. Define $\|f\|_L := \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|}$.

   a. Show that $\mathrm{Lip}_0([0, 1])$ is a vector space and $\|\cdot\|_L$ satisfies $(i) - (iii)$ in the definition of a norm.

   b. Show that $\|\cdot\|_L$ is NOT a norm.

   c. Define $\mathrm{Lip}_0([0, 1]) := \{f \in \mathrm{Lip}([0, 1]) : f(0) = 0\}$. Show that $\mathrm{Lip}_0([0, 1])$ is a vector subspace of $\mathrm{Lip}([0, 1])$ and that $\|\cdot\|_L$ is a norm on $\mathrm{Lip}_0([0, 1])$.

5. Consider the space of Riemann integrable functions $\mathcal{R}([0, 1])$. Define $\|f\|_1 := \int_0^1 |f|$ for all $f \in \mathcal{R}([0, 1])$.

   a. Show that $(\mathcal{C}([0, 1]), \|\cdot\|_1)$ is a normed space.

   b. Is $\|\cdot\|_1$ a norm on $\mathcal{R}([0, 1])$?

More Quick Practice

1. Let X be a normed space. We say that a subset $C \subset X$ is convex if for all $x, y \in C$ we have $tx+(1-t)y \in C$ for all $t \in [0,1]$. Define $B(x,r) := \{y \in X : \|y-x\| < r\}$ for all $x \in X$ and $r > 0$.

   a. Show that $B(0,1)$ is convex.

   b. Show that $B(x,r)$ is convex for all $x \in X$ and $r > 0$

2. Let $p \geq 1$. Define for all $f \in \mathcal{C}([0,1])$ that $\|f\|_p := (\int_0^1 |f|^p)^{1/p}$.

   a. Show that $\|\cdot\|_p$ satisfies (i), (ii), (iv) in the definition of a norm for all $p \geq 1$.

   b. When $p = 1, 2$, show that $\|\cdot\|_p$ is a norm.

   c. (Challenging, cf Tutorial 3). Show that $\|\cdot\|_p$ is a norm for all $p \geq 1$.

3. Let X be a normed space. Let (x_n) be a sequence in X. We say that

   - (x_n) is a Cauchy sequence in X if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|x_n - x_m\| < \epsilon$.
   - (x_n) converges to $x \in X$ if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|x_n - x\| < \epsilon$. In other words, $\lim_n \|x_n - x\| = 0$.

   Now consider $X = (\mathcal{C}([0,1]), \|\cdot\|_\infty)$. Let (f_n) be a Cauchy sequence in X.

   a. Show that for all $t \in [0,1]$, the sequence $(f_n(t))$ is a Cauchy sequence in $\mathbb{R}$.

   b. Following (a), define $f(t) := \lim_n f_n(t)$ for all $t \in [0,1]$. Show that (f_n) converges to f.

   c. (Tricky). Consider now instead $Y := (\mathcal{C}([0,1]), \|\cdot\|_1)$, where $\|f\|_1 := \int_0^1 |f|$. Show that not every Cauchy sequence in Y converges.
   (Hint: Find a sequence (f_n) in $\mathcal{C}([0,1])$ and $f \in \mathcal{R}([0,1]) \backslash \mathcal{C}([0,1])$ such that $\lim \|f_n - f\|_1 = 0$.

4. Let X be a normed space. We say that $f : X \to \mathbb{R}$ is continuous at $x \in X$ if for all $\epsilon > 0$, there exists $\delta > 0$ such that $\|x - y\| < \delta$ would imply $|f(x) - f(y)| < \epsilon$.

   a. Show that a linear functional $f : X \to \mathbb{R}$ is continuous for all $x \in X$ if and only if f is continuous at $0 \in X$.

   b. Suppose $X = (\mathbb{R}, |\cdot|)$. Show that every linear functional is continuous.

5. This question follows Q4. Let X be a normed space. Then we denote X^* the space of *continuous* linear functionals, that is, $f \in X^*$ if and only if f is continuous and linear.

   a. Show that a linear functional $f \in X^*$ if and only if there exists $M > 0$ such that $|f(x)| \leq M\|x\|$

   b. Define for all $f \in X^*$ that $\|f\|_{X^*} := \sup_{\|x\| \leq 1} |f(x)|$. Show that $(X^*, \|\cdot\|_{X^*})$ is a normed space.

   c. (Challenging). Show that $(X^*, \|\cdot\|_{X^*})$ satisfies the Cauchy criteria, that is, every Cauchy sequence in X^* converges to some element in X^*.

6. Let X, Y be a normed space. We say that $T : X \to Y$ is an isometry if $\|Tx\|_Y = \|x\|_X$ for all $x \in X$.

   a. Show that every isometry is injective.

   b. For $f \in \mathcal{C}^1([0,1])$, define $\|f\|_L := \sup_{x \neq y} \frac{|f(x)-f(y)|}{|x-y|}$ and $\|f\|_S := \|f'\|_\infty = \sup_{t \in [0,1]} |f'(t)|$. Show that $\|\cdot\|_L$ and $\|\cdot\|_S$ satisfies (i) to (iii) in the definition of a norm but is not a norm on $\mathcal{C}^1([0,1])$

   c. Consider the subspace $Z := \mathcal{C}_0^1([0,1]) := \{f \in \mathcal{C}(0,1) : f(0) = 0\}$. Show that $(Z, \|\cdot\|_L)$ and $(Z, \|\cdot\|_S)$ are normed spaces.

   d. Consider $X := (Z, \|\cdot\|_L)$ and $Y := (Z, \|\cdot\|_S)$. Show that there exists a *surjective linear* isometry, or a linear isometric isomorphism, between X and Y.

7. (Functional Interpretation of FTC, cf. Tutorial 7). Let $X := (\mathcal{C}([0,1]), \|\cdot\|_\infty)$ and $Y := (\mathcal{C}_0^1([0,1]), \|\cdot\|_L)$, which is defined in Q6. Define $T : X \to Y$ by $Tf(t) := \int_0^t f$ for all $t \in [0,1]$. Show that T is a linear isometric isomorphism between X, Y by finding explicitly the inverse of T.