Unless otherwise specified, if we write (a,b) or $[a,b]$, it is always the case that $a < b \in \mathbb{R}$.

1 Revisiting Convexity

Definition 1.1. Let $f : (a,b) \to \mathbb{R}$ be a function. Let $c \in (a,b)$

- We say that f has a right-hand derivative at $c \in (a,b)$ if $f'_+(c) := \lim_{x\to c^+} \frac{f(x)-f(c)}{x-c}$ exists.
- We say that f has a left-hand derivative at $c \in (a,b)$ if $f'_-(c) := \lim_{x\to c^-} \frac{f(x)-f(c)}{x-c}$ exists.

Quick Practice

0. Let $f : (a,b) \to \mathbb{R}$. Show that f is differentiable at $c \in (a,b)$ if and only if f has an equal right-hand and left-hand derivative at c, that is, $f'_+(c) = f'_-(c)$.

   Solution. Covered in Tutorial.

1. Let $f : I := (a,b) \to \mathbb{R}$. Show that f is convex if and only if for all $x < y < z \in I$, we have

$$\frac{f(x)-f(y)}{x-y} \le \frac{f(z)-f(y)}{z-y}$$

   Solution. Covered in Tutorial (cf. solutions in Tutorial 1).

2. Let $f : I := (a,b) \to \mathbb{R}$ be a function. Let $p \in I$. Define $Q_p(x) := \frac{f(x)-f(p)}{x-p}$ for all $x \neq p \in I$.

   (a) Show that f is convex if and only if for all $p \in I$ and $x \le y \in I\backslash\{p\}$ we have $Q_p(x) \le Q_p(y)$, that is, Q_p is increasing on $I\backslash\{p\}$

   (b) Using (a), show that if f is convex, then f has left and right derivative at every $p \in I$.

   (c) Hence, show that if f is convex, then for all $x < y \in I$, we have

$$f'_-(x) \le f'_+(x) \le f'_-(y) \le f'_+(y)$$

   Solution. Covered in Tutorial.

3. Let $f : I := (a,b) \to \mathbb{R}$ be a differentiable function.

   (a) Show that f is convex if and only if f' is increasing.

   (b) Show that every convex differentiable function on I is continuously differentiable.

   Solution. (a). Make use of Q2a and MVT. (b). This follows from HW2 Q2 that existence of left and right limits of a derivative would imply respectively left and right continuity.

4. Let $f : (0,1) \to \mathbb{R}$ be a convex function.

   (a) Show that f is continuous.

   (b) Can f be nowhere differentiable? Give an example or prove your assertion otherwise.

   Solution. (a). Write $f(x) = \frac{f(x)-f(y)}{x-y}(x-y) + f(y)$ for all $x \neq y$. The result follows by considering left and right derivatives. (b). No. A convex function has at most countable non-differentiable point. The solution is similar to why monotone functions only have at most countably many discountinuity. The precise solution is covered in Tutorial.

5. Let $I := (a, b)$. We say $g : I \to \mathbb{R}$ an affine function if $g(x) := Mx + c$ for some $M, c \in \mathbb{R}$.

   (a) Let L be a collection of affine functions on I. Define $\phi(x) := \sup\{f(x) : f \in L\}$ for all $x \in I$. Suppose ϕ is well-defined, that is, $\phi(x) \in \mathbb{R}$ for all $x \in I$. Show that ϕ is a convex function.

   (b) Let $f : I \to \mathbb{R}$ be convex differentiable. Show that there exists a collection of affine functions T such that $f(x) := \sup\{L(x) : L \in T\}$.

   (c) Let $f : I \to \mathbb{R}$ be convex (not necessarily differentiable). Is the conclusion of part (b) still true? Prove your assertion or provide a counter-example.

   Solution. (a) is easy. (b). Note that $f(x) \geq f'(c)(x - c) + f(c)$ for all $x \neq c$. (c). Yes. Replace the derivative in (b) by either the left or the right derivative. A pictorial interpretation is covered in Tutorial.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded function. Let $\phi : \mathbb{R} \to \mathbb{R}$ be convex. Suppose f is Riemann integrable over every compact interval $[a, b]$.

   (a) Show that $\phi(\int_0^1 f(t)dt) \leq \int_0^1 \phi(f(t))dt$. This is called the Jensen's inequality.

   (b) Using (a), show that for all $0 < p \leq q$ and $f \in \mathcal{R}([a, b])$ we have $(\int_0^1 |f|^p)^{1/p} \leq (\int_0^1 |f|^q)^{1/q}$

   (c) Is part (a) still true if the domain of integration is changed, that is, do we have $\phi(\int_a^b f) \leq \int_a^b \phi \circ f$ for any $a < b \in \mathbb{R}$? Prove your assertion or provide a counter-example.

   Solution. (a). Use Q5, covered in Tutorial. (b). Note that $q/p \geq 1$ and $x \mapsto |x|^\alpha$ is a convex function on $\mathbb{R}$. (c). No. Consider $\phi(t) := t^2, f(x) := x$ and $a := 0$. Then solve the inequality $(\int_0^b x dx)^2 > \int_0^b x^2 dx$. It follows that we have

$$\frac{1}{4}b^4 = (\frac{1}{2}b^2)^2 > \frac{1}{3}b^3$$

   which is true as long as $b > \frac{4}{3}$

7. Let $X : \mathbb{R} \to \mathbb{R}$ be continuous. Define $\text{var}(X) := \int_0^1 X^2 - (\int_0^1 X)^2$.

   (a) Show that $\text{var}(X) \geq 0$.

   (b) Show that $\text{var}(X - c) = \text{var}(X)$ for all $c \in \mathbb{R}$.

   (c) Define $\mu := \int_0^1 X$. Show that $\text{var}(X) = 0$ if and only if $X(\omega) = \mu$ for all $\omega \in [0, 1]$, that is, X is constantly μ.

   (d) Suppose $m \leq X \leq M$ for some $m, M \in \mathbb{R}$ point-wise. Show that $\text{var}(X) \leq \frac{(M-m)^2}{4}$

   Solution. (a). By Jensen's inequality using $\phi(t) := t^2$, it follows that we have $(\int_0^1 X)^2 \leq \int_0^1 X^2$.
   (b). Note that $\int_0^1 (X - c)^2 = \int_0^1 X^2 - 2c \int_0^1 X + c^2$, in which the last term has made use of the fact that the domain of integration is of length 1. Moreover, we have $(\int_0^1 X - c)^2 = (\int_0^1 X)^2 - 2c \int_0^1 X + c^2$. The result follows by considering their difference.
   (c). $(\Rightarrow)$. By (b), we have $\text{var}(X - \mu) = 0$. Note that $\int_0^1 X - \mu = 0$. Hence, $\text{var}(X - \mu) = \int_0^1 (X - \mu)^2$. Therefore we have $\int_0^1 (X - \mu)^2 = 0$ by assumption. Note that $(X - \mu)^2$ is non-negative continuous. It follows that $(X - \mu)^2 = 0$ pointwise (cf. HW5 Q2) and so $X = \mu$ constantly.
   (d). We first consider the case that $-d \leq X \leq d$ for $d > 0$. Then $0 \leq \int_0^1 X^2 \leq d^2$. In addition we have $-d \leq \int_0^1 X \leq d$ and so $0 \leq (\int_0^1 X)^2 \leq d^2$. Therefore $\text{var}(X) \leq d^2 - 0 = \frac{(d-(-d))^2}{4}$. For arbitrary $m \leq M$. Note that take $r := \frac{M+m}{2}$. Then $M - r = \frac{M-m}{2}$ and $m - r = \frac{m-M}{2}$. Hence, we have $-(M - m)/2 \leq X - r \leq (M - m)/2$. It follows from the previous case and part (b) that $\text{var}(X) = \text{var}(X - r) \leq (\frac{M-m}{2})^2$

2 Approximations in Integration

Proposition 2.1. *Let $f \in \mathcal{R}([0,1])$. Then for all partitions $P \subset [0,1]$, there exists a step function s_P such that $\int_0^1 s_P = L(f,P)$. Furthermore, we can choose the collection $\{s_P\}$ such that $s_Q \geq s_P$ whenever $Q \succeq P$, that is, whenever Q refines P.*

Proof. Let $P := \{x_i\}_{i=1}^k$ be a partition over $[0,1]$. Then $L(f,P) = \sum_{i=1}^k m_i(f,P)(x_{i-1},x_i)$. Now we define a step function s_P by $s_P \equiv m_i(f,P)$ on (x_{i-1},x_i). Note that there is a unique way of defining s_P on the end-points $P \subset [0,1]$ such that s_P is right-continuous on $[0,1)$ and left-continuous at 1. We define s_P on the end-points according to that. Then it is clear that by splitting domains (or by Lecture Theorems), we have $\int_0^1 s_P = L(f,P)$. Furthermore, it is not hard it see that for a refinement $Q \supset P$, we have $s_Q \geq s_P$ point-wise everywhere. □

Example 2.2 (cf. Tutorial 6; motivated from 1920 Home Test 1 Q2)**.** *Recall that a function $s : [0,1] \to \mathbb{R}$ is a step function over $[0,1]$ if there exists a partition $P := \{x_i\}_{i=0}^k \subset [0,1]$ such that s is constant over (x_{i-1},x_i).*

(a) Let $f \in \mathcal{R}([0,1])$. Show that there exists a sequence of step functions (s_n) over $[0,1]$ such that $s_n \leq s_{n+1}$ pointwise for all $n \in \mathbb{N}$ and $\lim_n \int_0^1 s_n = \int_0^1 f$.

(b) Let $f \in \mathcal{C}([0,1])$, that is f is continuous. Show that there exists a sequence of step functions (s_n) uniformly approximating f, that is, $\lim_n \sup_{x\in[0,1]} |s_n(x) - f(x)| = 0$. Hence, show that the sequence also satisfies $\lim_n \int_0^1 s_n = \int_0^1 f$.

(c) Suppose $f \in \mathcal{R}([0,1])$. Is it always true that f is uniformly approximated by step functions, that is, can the assumption in (b) be relaxed to only integrable functions?

Solution. a. Let (ϵ_n) be a sequence of decreasing positive number such that $\epsilon_n \downarrow 0$. By considering lower integral, there exists a sequence of partitions (P_n) such that $\int_0^1 f - \epsilon_n < L(f,P_n)$. Now define $Q_n := \bigcup_{i=1}^n P_i$. Then it is not hard to see that (Q_n) is increasing with respect to refinements and we have $\int_0^1 f - \epsilon_n < L(f,Q_n)$. Therefore the step functions defined by $s_n := s_{Q_n}$ according to the way stated in the beginning is point-wise increasing. Furthermore we have $\int_0^1 f - \epsilon_n < \int_0^1 s_n = L(f,Q_n)$. This clearly implies that $\lim_n \int_0^1 s_n = \int_0^1 f$ as $n \to \infty$.

b. Let (ϵ_n) be a sequence of decreasing positive number such that $\epsilon_n \downarrow 0$. By compactness, f is uniformly continuous. Hence, there exists $\delta_n > 0$ such that $|f(x) - f(y)| < \epsilon_n$ when $|x-y| < \delta_n$ for all $n \in \mathbb{N}$. For all $n \in \mathbb{N}$ choose a partition $P_n := \{x_i^n\}_{i=1}^k \subset [0,1]$ such that $\max_{i=1}^k |x_i^n - x_{i-1}^n| < \delta_n$. Define s_n to be some step function such that s_n is right continuous on $[0,1)$ and left continuous at 1 such that $s_n \equiv c_{i,n}$ on (x_{i-1}^n, x_i^n) for some $c_{i,n} \in (x_{i-1}^n, x_i^n)$. It follows clearly that $\sup_{x\in[0,1]} |s_n(x) - f(x)| \leq \epsilon_n$ for all $n \in \mathbb{N}$. Hence, (s_n) approximates f uniformly.

Next we show that $\int_0^1 s_n \to \int_0^1 f$. This follows because we have for all $n \in \mathbb{N}$ that

$$\left|\int_0^1 s_n - \int_0^1 f\right| = \left|\int_0^1 (s_n - f)\right| \leq \int_0^1 |s_n - f| \leq \sup_{x\in[0,1]} |s_n(x) - f(x)|$$

c. No. We claim that that the indicator function $f := \mathbb{1}_{\{\frac{1}{n}:n\in\mathbb{N}\}}$ cannot be uniformly approximated by step functions. Suppose not. Then there exists a step function $s := \sum_{i=1}^k c_i \mathbb{1}_{(x_i,x_{1-i})}$ where $c_i \in \mathbb{R}$ and $\{x_i\} \subset P$ is a partition such that $\sup_{x\in[0,1]} |f(x) - s(x)| < \frac{1}{3}$. It follows that $|f(x) - s(x)| \leq \frac{1}{3}$ for all $x \in (0 =: x_0, x_1)$. In other words, we have $|c_1 - f(x)| \leq \frac{1}{3}$ for all $x \in (0,x_1)$. Nonetheless, note that f attains both 0 and 1 infinitely over $(0,x_1)$. Therefore contradiction arises as it cannot happen at the same time that $|c_1| \leq \frac{1}{3}$ and $|c_1 - 1| \leq \frac{1}{3}$ by the triangle inequality.