Unless otherwise specified, if we write (a, b) or $[a, b]$, it is always the case that $a < b \in \mathbb{R}$.

1 The Fundamental Theorem of Calculus

Theorem 1.1. *Let $f \in \mathcal{C}([a, b])$. Define the function $F(t) := \int_a^t f$ for all $t \in [a, b]$.*

i. Then F is continuous on $[a, b]$ and differentiable on (a, b) with $F' = f$ on (a, b).

ii. Furthermore $F(y) - F(x) = \int_x^y f$ for all $x \le y \in [a, b]$.

Remark. (i) may not be true if $f \in \mathcal{R}([a, b])$. (ii) is still true as long as $f \in \mathcal{R}([a, b])$ and F is an anti-derivative of f, that is, f is some function satisfying (i).

Conceptual Practice

1. Let $f \in \mathcal{R}([a, b])$. Define $F(t) := \int_a^t f$ for all $t \in [a, b]$.

   (a) Show that F is a Lipschitz function on $[a, b]$.

   (b) Suppose f is continuous at $c \in [a, b]$. Show that F is differentiable at $c \in [a, b]$

   (c) Let f be increasing on $[a, b]$. Show that there exists a Lipschitz function F and a countable set $C \subset [a, b]$ such that $F' = f$ on $[a, b] \backslash C$.

   Solution. Covered in the Tutorial.

2. Let $G(x) := x^2 \cos(1/x)$ for $x \neq 0$ and $G(0) := 0$ for all $x \in \mathbb{R}$.

   (a) Show that G is differentiable and compute its derivative.

   (b) Define $F(t) := \int_0^t \sin(1/x)dx$ for all $t \in \mathbb{R}$. Show that F is a differentiable function..

   Solution. (a). $G'(x) = 2x\cos(1/x) + \sin(1/x)$ for all $x \neq 0$ by product rule and $G'(0) = 0$.
   (b). Define $h(x) := 2x\cos(1/x)$ for $x \neq 0$ and $h(0) := 0$. Then h is continuous by sandwich theorem. Define $H(t) := \int_0^t h(x)dx$ for all $t \in \mathbb{R}$. Then H is differentiable with $H' = h$ by FTC. Define $f(x) := \sin(1/x)$ for $x \neq 0$ and $f(0) := 0$. Note that $f(x) = G'(x) - h(x)$ for all $x \in \mathbb{R}$. Since G', h is Riemann integrable, it follows that $F(t) = \int_0^t f(x)dx = \int_0^t G'(x) + h(x)dx = G(t) - H(t)$ for all $t \in \mathbb{R}$. Since G, H are differentiable, it follows that F is differentiable.

3. Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable.

   (a) Show that $F(x) - F(y) = \int_y^x f$ for all $x \le y \in [a, b]$ if $F : [a, b] \to \mathbb{R}$ is differentiable on (a, b) with $F' = f$ and continuous on $[a, b]$.

   (b) Give a shorter proof for part (i) if f is also continuous.

   Solution. (a). By MVT; covered in lecture notes. (b). Note that $G(t) := \int_a^t f(x)dx$ is differentiable by FTC. Furthermore, $G' = F' = f$. Hence, $F = G + C$ for some constant $C \in \mathbb{R}$. The equality of part (a) follows clearly.

4. Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuously differentiable functions. Show that for all $a < b \in \mathbb{R}$

$$f(b)g(b) - f(a)g(a) = \int_a^b fg' + \int_a^b f'g$$

   Solution. Note that $(fg)'$ is continuous by assumption. The result follows from FTC and product rule.

5. (Alternative viewpoint on FTC, modified). For all Lipschitz functions $f : [0,1] \to \mathbb{R}$, we define the constant $\|f\|_L := \sup_{x \neq y} \left|\frac{f(x)-f(y)}{x-y}\right|$. Furthermore, for all continuous functions $f : [0,1] \to \mathbb{R}$, we define the constant $\|f\|_\infty := \sup_{x\in[0,1]} |f(x)|$.

   (a) Let $f \in \mathcal{C}^1([0,1])$. Show that $\|f\|_L = \|f'\|_\infty$. Hence, show that $\|f\|_L = 0$ if and only if f is a constant function if $f \in \mathcal{C}^1([0,1])$.

   (b) Let $\mathcal{C}_0^1([0,1]) := \{f \in \mathcal{C}^1([0,1]) : f(0) = 0\}$. Show that $\mathcal{C}_0^1([0,1])$ is a vector subspace of $\mathcal{C}^1([0,1])$ with the property that if $f \in \mathcal{C}_0^1([0,1])$ then $f = 0$ if and only if $\|f\|_L = 0$. We call $\mathcal{C}_0^1[0,1]$ the space of pointed $\mathcal{C}^1$ maps.

   (c) Let $T : \mathcal{C}[0,1] \to \mathcal{C}_0^1[0,1]$ be defined by $Tf(t) := \int_0^t f$ for all $t \in [0,1]$ and $f \in \mathcal{C}[0,1]$.

      i. Show that T is a well-defined linear map between vector spaces.

      ii. Show that T is a linear isomorphism by explicitly finding the inverse for T.

      iii. Show that T is a linear isometric isomorphism, that is, T is a linear isomorphism and for all $f \in \mathcal{C}([0,1])$, we have $\|f\|_\infty = \|Tf\|_L$

      iv. Is T invertible if we consider the codomain to be $\mathcal{C}^1([0,1])$ instead of the space of pointed maps?

   Solution. (a) and (b) are easy and so whose solutions are omitted. (c). (i) - (ii) follows from FTC. (iii). Note that f is the derivative of Tf. It follows from Tutorial 2 Q4 that $\|f\|_\infty = \|(Tf)'\|_\infty = \|Tf\|_L$, which is not hard to show. (iv). No, because by the definition of T, we always have $Tf(0) = 0$ for all $f \in \mathcal{C}([0,1])$. Nonetheless, it is not always the case that $g(0) = 0$ for $g \in \mathcal{C}^1([0,1])$ and so T is not surjective.

6. (Riemann–Stieltjes integral) Let $f : [0,1] \to \mathbb{R}$ be a bounded function and $g : [0,1] \to \mathbb{R}$ be an increasing function. Let $P \subset [0,1]$ be a partition. We define

   - the upper sum $U(f,P,g) := \sum_{i=1}^k M_i(f,P)(g(x_i) - g(x_{i-1}))$
   - and the lower sum $U(f,P,g) := \sum_{i=1}^k m_i(f,P)(g(x_i) - g(x_{i-1}))$

   It is not hard to see that $(U(f,P,g))_{P\subset[0,1]}$ is a bounded below decreasing net and $(L(f,P,g))_{P\subset[0,1]}$ is a bounded above increasing net with respect to refinements. Therefore, similar to the Darboux case, we can define $\overline{\int_0^1} fdg := \lim_P U(f,P,g) = \inf_P U(f,P,g)$ and $\underline{\int_0^1} fdg := \lim_P L(f,P,g) = \sup L(f,P,g)$. These are called the Riemann-Stieltjes upper and lower intergrals of f with respect to g respectively. We say that f is Riemann-Stieltjes integrable with repsect to g if $\overline{\int_0^1} fdg = \underline{\int_0^1} fdg$; we write the Riemann-Stieltjes integral as $\int_0^1 fdg$

   (a) Verify that the upper and lower Riemann-Stieltjes integrals are well-defined.

   (b) Show that f is R-S (Riemann-Stieltjes) intergrable with respect to g is and only if for all $\epsilon > 0$ there exists a partition $P \subset [0,1]$ such that $U(f,P,g) - L(f,P,g) < \epsilon$.

   (c) Suppose f is R-S integrable with respect to g and g is continuously differentiable. Suppose further that $fg' \in \mathcal{R}([0,1])$. Show that $\int_0^1 fdg = \int_0^1 fg'$ where the right-hand side is the ordinary intergral.

   Solution. (a), (b) are similar to the case of Darboux integrals and so are omitted.
   (c). It suffices to show that for all $\epsilon > 0$, there exists a partition $P \subset [0,1]$ that $\left|\int_0^1 fg' - U(f,P,g)\right| < \epsilon$. To this end. let $\epsilon > 0$. Then as $fg', g' \in \mathcal{R}([0,1])$, there exists a (largely refined) partition P such that $\left|U(fg',P) - \int_0^1 fg'\right| < \epsilon$, $\sum_i \omega_i(gf',P)\Delta x_i < \epsilon$ and $\sum_i \omega_i(g,P)\Delta x_i < \epsilon$. Next, we claim that

   $$|\sup fg'(I_i) - \sup(f(I_i))g'(x)| \leq \omega_i(fg',P) + \omega_i(g',P) \sup |f|([0,1])$$

   for all $x \in I_i$ and I_i an interval component of P. To show the claim, let (x_n) and (y_n) be such that $fg'(x_n) \to \sup fg'(I_i)$ and $f(y_n) \to \sup f(I_i)$. Then we have for all $n \in \mathbb{N}$ that

   $$|fg'(x_n) - f(y_n)g(x)| \leq |fg'(x_n) - fg'(y_n)| + |f(y_n)||g(y_n) - g(x)| \leq \omega_i(fg',P) + \omega_i(g',P) \sup |f|([0,1])$$

   The claim follows as $n \to \infty$. With the help of the claim as well as MVT on g, it is then not hard to see that we have the approximation $|U(fg',P) - U(f,P,g)| \leq \sum_i \omega_i(fg',P) + \omega_i(g',P) \sup |f|([0,1]) \leq (1 + \sup f([0,1]))\epsilon$. The result follows clearly.

2 Riemann Sum

Theorem 2.1. *Let $f : [a, b] \to \mathbb{R}$ be bounded. Then $f \in \mathcal{R}([a, b])$ if and only if there exists $A \in \mathbb{R}$ such that for all $\epsilon > 0$, there exists $\delta > 0$ such that for all partitions $P := \{x_i\}_{i=0}^k \subset [a, b]$ with $\|P\| := \max_{i=1}^k |x_i - x_{i-1}| < \delta$ and for all tags $(\xi_i)_{i=1}^k$, that is, $\xi_i \in [x_{i-1}, x_i]$, we have*

$$\left| \sum_{i=1}^{k} f(\xi_i)(x_i - x_{i-1}) - A \right| < \epsilon$$

In that case, we in fact have $A = \int_a^b f$. We denote $R(f, P, \{\xi_i\}) := \sum_{i=1}^k f(\xi_i)(x_i - x_{i-1})$ the Riemann sum with respect to the partition P and tags $\{x_i\}$.

Quick Practice

1. Let $F : \mathbb{R} \to \mathbb{R}$ be continuously differentiable. Using Riemann sums, show that $F(a) - F(b) = \int_b^a F'$.
   Solution. From the textbook.

2. Show using Riemann sums that $\left|\int_a^b f\right| \leq M(b - a)$ if $f \in \mathcal{R}([a, b])$ and $|f| \leq M$ for $M > 0$.
   Solution. From the textbook.

3. Evaluate the following limits
   (a) $\lim_n \sum_{k=1}^{n} \frac{n}{n^2 + k^2}$
   (b) $\lim_n \frac{1}{n}(\sin\left(\frac{\pi}{n}\right) + \sin\left(\frac{2\pi}{n}\right) + \cdots + \sin(\pi))$
   (c) $\lim_n (\frac{n+1}{n^2} + \frac{n+2}{n^2} + \cdots + \frac{2n}{n^2})$

   Solution. Write the sums into the form $\sum_{k=1}^{n} \frac{1}{n} f(\frac{k}{n})$ where $f \in \mathcal{R}([0, 1])$. Then it follows from the Riemann sum characterization that $\lim_n \sum_{k=1}^{n} \frac{1}{n} f(\frac{k}{n}) = \int_0^1 f(t)dt$.

4. Show that $\int_0^1 x^2 = 1/3$ by considering Riemann sums.
   Solution. Consider limits similar to those in Q3.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be decreasing. Show that for all $N > 1 \in \mathbb{N}$, we have

$$\int_1^{N+1} f \leq \sum_{k=1}^{N} f(k) \leq \int_0^N f$$

   Solution. Consider suitable upper and lower sums.

6. Show that for all $n > 2 \in \mathbb{N}$, we have

$$n \sum_{k=n}^{\infty} \frac{1}{k^2} \leq 2$$

   Solution. Use Q5 and take limits.