Unless otherwise specified, if we write (a, b) or $[a, b]$, it is always the case that $a < b \in \mathbb{R}$.

1 The Fundamental Theorem of Calculus

Theorem 1.1. *Let $f \in \mathcal{C}([a, b])$. Define the function $F(t) := \int_a^t f$ for all $t \in [a, b]$.*

i. Then F is continuous on $[a, b]$ and differentiable on (a, b) with $F' = f$ on (a, b).

ii. Furthermore $F(y) - F(x) = \int_x^y f$ for all $x \leq y \in [a, b]$.

Remark. (i) may not be true if $f \in \mathcal{R}([a, b])$. (ii) is still true as long as $f \in \mathcal{R}([a, b])$ and F is an anti-derivative of f, that is, f is some function satisfying (i).

Conceptual Practice

1. Let $f \in \mathcal{R}([a, b])$. Define $F(t) := \int_a^t f$ for all $t \in [a, b]$.
   (a) Show that F is a Lipschitz function on $[a, b]$.
   (b) Suppose f is continuous at $c \in [a, b]$. Show that F is differentiable at $c \in [a, b]$
   (c) Let f be increasing on $[a, b]$. Show that there exists a Lipschitz function F and a countable set $C \subset [a, b]$ such that $F' = f$ on $[a, b] \backslash C$.

2. Let $G(x) := x^2 \cos(1/x)$ for $x \neq 0$ and $G(0) := 0$ for all $x \in \mathbb{R}$.
   (a) Show that G is differentiable and compute its derivative.
   (b) Define $F(t) := \int_0^t \sin(1/x)dx$ for all $t \in \mathbb{R}$. Show that F is a differentiable function..

3. Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable.
   (a) Show that $F(x) - F(y) = \int_y^x f$ for all $x \leq y \in [a, b]$ if $F : [a, b] \to \mathbb{R}$ is differentiable on (a, b) with $F' = f$ and continuous on $[a, b]$.
   (b) Give a shorter proof for part (i) if f is also continuous.

4. Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuously differentiable functions. Show that for all $a < b \in \mathbb{R}$

$$f(b)g(b) - f(a)g(a) = \int_a^b fg' + \int_a^b f'g$$

5. (Alternative viewpoint on FTC, modified). For all Lipschitz functions $f : [0,1] \to \mathbb{R}$, we define the constant $\|f\|_L := \sup_{x \neq y} \left| \frac{f(x)-f(y)}{x-y} \right|$. Furthermore, for all continuous functions $f : [0,1] \to \mathbb{R}$, we define the constant $\|f\|_\infty := \sup_{x \in [0,1]} |f(x)|$.

   (a) Let $f \in \mathcal{C}^1([0,1])$. Show that $\|f\|_L = \|f'\|_\infty$. Hence, show that $\|f\|_L = 0$ if and only if f is a constant function if $f \in \mathcal{C}^1([0,1])$.

   (b) Let $\mathcal{C}_0^1([0,1]) := \{f \in \mathcal{C}^1([0,1]) : f(0) = 0\}$. Show that $\mathcal{C}_0^1([0,1])$ is a vector subspace of $\mathcal{C}^1([0,1])$ with the property that if $f \in \mathcal{C}_0^1([0,1])$ then $f = 0$ if and only if $\|f\|_L = 0$. We call $\mathcal{C}_0^1[0,1]$ the space of pointed $\mathcal{C}^1$ maps.

   (c) Let $T : \mathcal{C}[0,1] \to \mathcal{C}_0^1[0,1]$ be defined by $Tf(t) := \int_0^t f$ for all $t \in [0,1]$ and $f \in \mathcal{C}[0,1]$.

      i. Show that T is a well-defined linear map between vector spaces.

      ii. Show that T is a linear isomorphism by explicitly finding the inverse for T.

      iii. Show that T is a linear isometric isomorphism, that is, T is a linear isomorphism and for all $f \in \mathcal{C}([0,1])$, we have $\|f\|_\infty = \|Tf\|_L$

      iv. Is T invertible if we consider the codomain to be $\mathcal{C}^1([0,1])$ instead of the space of pointed maps?

6. (Riemann–Stieltjes integral) Let $f : [0,1] \to \mathbb{R}$ be a bounded function and $g : [0,1] \to \mathbb{R}$ be an increasing function. Let $P \subset [0,1]$ be a partition. We define

   - the upper sum $U(f,P,g) := \sum_{i=1}^k M_i(f,P)(g(x_i) - g(x_{i-1}))$
   - and the lower sum $U(f,P,g) := \sum_{i=1}^k m_i(f,P)(g(x_i) - g(x_{i-1}))$

   It is not hard to see that $(U(f,P,g))_{P \subset [0,1]}$ is a bounded below decreasing net and $(L(f,P,g))_{P \subset [0,1]}$ is a bounded above increasing net with respect to refinements. Therefore, similar to the Darboux case, we can define $\overline{\int_0^1} f dg := \lim_P U(f,P,g) = \inf_P U(f,P,g)$ and $\underline{\int_0^1} f dg := \lim_P L(f,P,g) = \sup L(f,P,g)$. These are called the Riemann-Stieltjes upper and lower intergrals of f with respect to g respectively. We say that f is Riemann-Stieltjes integrable with repsect to g if $\overline{\int_0^1} f dg = \underline{\int_0^1} f dg$; we write the Riemann-Stieltjes integral as $\int_0^1 f dg$

   (a) Verify that the upper and lower Riemann-Stieltjes integrals are well-defined.

   (b) Show that f is R-S (Riemann-Stieltjes) intergrable with respect to g is and only if for all $\epsilon > 0$ there exists a partition $P \subset [0,1]$ such that $U(f,P,g) - L(f,P,g) < \epsilon$.

   (c) Suppose f is R-S integrable with respect to g and g is continuously differentiable. Suppose further that $fg' \in \mathcal{R}([0,1])$. Show that $\int_0^1 f dg = \int_0^1 fg'$ where the right-hand side is the ordinary intergral.

2 Riemann Sum

Theorem 2.1. *Let $f : [a,b] \to \mathbb{R}$ be bounded. Then $f \in \mathcal{R}([a,b])$ if and only if there exists $A \in \mathbb{R}$ such that for all $\epsilon > 0$, there exists $\delta > 0$ such that for all partitions $P := \{x_i\}_{i=0}^k \subset [a,b]$ with $\|P\| := \max_{i=1}^k |x_i - x_{i-1}| < \delta$ and for all tags $(\xi_i)_{i=1}^k$, that is, $\xi_i \in [x_{i-1}, x_i]$, we have*

$$\left| \sum_{i=1}^{k} f(\xi_i)(x_i - x_{i-1}) - A \right| < \epsilon$$

In that case, we in fact have $A = \int_a^b f$. We denote $R(f, P, \{\xi_i\}) := \sum_{i=1}^k f(\xi_i)(x_i - x_{i-1})$ the Riemann sum with respect to the partition P and tags $\{x_i\}$.

Quick Practice

1. Let $F : \mathbb{R} \to \mathbb{R}$ be continuously differentiable. Using Riemann sums, show that $F(a) - F(b) = \int_b^a F'$.

2. Show using Riemann sums that $\left|\int_a^b f\right| \leq M(b-a)$ if $f \in \mathcal{R}([a,b])$ and $|f| \leq M$ for $M > 0$.

3. Evaluate the following limits

   (a) $\lim_n \sum_{k=1}^{n} \frac{n}{n^2 + k^2}$

   (b) $\lim_n \frac{1}{n}(\sin\left(\frac{\pi}{n}\right) + \sin\left(\frac{2\pi}{n}\right) + \cdots + \sin(\pi))$

   (c) $\lim_n (\frac{n+1}{n^2} + \frac{n+2}{n^2} + \cdots + \frac{2n}{n^2})$

4. Show that $\int_0^1 x^2 = 1/3$ by considering Riemann sums.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be decreasing. Show that for all $N > 1 \in \mathbb{N}$, we have

$$\int_1^{N+1} f \leq \sum_{k=1}^{N} f(k) \leq \int_0^N f$$

6. Show that for all $n > 2 \in \mathbb{N}$, we have

$$n \sum_{k=n}^{\infty} \frac{1}{k^2} \leq 2$$