Unless otherwise specified, if we write (a, b) or $[a, b]$, it is always the case that $a < b \in \mathbb{R}$.

1 Basic Results on Riemann Integrability

Definition 1.1. Let $f : [a, b] \to \mathbb{R}$ be bounded. Then we say that f is (Riemann) integrable, or $f \in \mathcal{R}([a, b])$ if $\overline{\int_a^b} f = \underline{\int_a^b} f$. Equivalently, for all $\epsilon > 0$, there exists a partition $P \subset [a, b]$ such that

$$U(f, P) - L(f, P) = \sum_{i=1}^{k} \omega_i(f, P)(x_i - x_{i-1}) < \epsilon$$

where $\omega_i(f, P) := \operatorname{diam} f([x_{i-1}, x_i]) := \sup_{x,y\in[x_{i-1},x_i]} |f(x) - f(y)|$. In such case, write $\int_a^b f := \overline{\int_a^b} f = \underline{\int_a^b} f$.

Conceptual Quick Practice

1. Let $f, g \in \mathcal{R}([a, b])$ and $\alpha \in \mathbb{R}$.
   (a) Show that $f + g \in \mathcal{R}([a, b])$ and $\int_a^b (f + g) = \int_a^b f + \int_a^b g$
   (b) Show that $\alpha f \in \mathcal{R}([a, b])$ and $\int_a^b \alpha f = \alpha \int_a^b f$
   (c) If $f \le g$ pointwise, then $\int_a^b f \le \int_a^b g$.

2. Let $f \in \mathcal{R}([a, b])$.
   (a) Show that $f^2 \in \mathcal{R}([a, b])$
   (b) Show that $fg \in \mathcal{R}([a, b])$ for all $g \in \mathcal{R}([a, b])$.

3. Let $f \in \mathcal{R}([a, b])$.
   (a) Suppose $g : [a, b] \to \mathbb{R}$ is a function such that $f = g$ for at most finitely many points. Show that $g \in \mathcal{R}([a, b])$.
   (b) Is the above true if "finitely many points" is replaced by "countably many points"?

4. Let $f \in \mathcal{R}([a, b])$ be bounded. Let $g : I \to \mathbb{R}$ be a function such that $f([a, b]) \subset I$ where I is some interval.
   (a) Show that $g \circ f \in \mathcal{R}([a, b])$ if g is Lipschitz continuous.
   (b) Is the above true if g is only continuous?

5. Show that $C([a,b])$, the space of continuous functions, is a vector subspace of $\mathcal{R}([a,b])$.

6. Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Let $c \in (a,b)$.

   (a) Show that $f \in \mathcal{R}[a,b]$ if and only if the restrictions are integrable, that is, we have both $f\mid_{[a,c]} \in \mathcal{R}([a,c])$ and $f\mid_{[c,b]} \in \mathcal{R}([c,b])$

   (b) Let $A \subset [a,b]$. Then we define $\mathbb{1}_A(x) := 1$ if $x \in A$ and zeros elsewhere. We call this the indicator function of A. A function f is said to be a *step function* if there exists a mutually disjoint finite collection of intervals $\{I_i\}_{i=1}^k$ such that f is a linear combination of $\{\mathbb{1}_{I_i}\}_{i=1}^k$, that is, $f = \sum_{i=1}^k a_i \mathbb{1}_{I_i}$ for some $\{a_i\}_{i=1}^k \subset \mathbb{R}$. Show that every step function is Riemann integrable.

7. Let $f(x) := \sin(1/x)$ for $x \in (0,1]$ and $f(0) := 0$. Show that $f \in \mathcal{R}([0,1])$.

8. This exercise provides an example showing the compositions of Riemann integrable functions may not be Riemann integrable.

   (a) Define $f(x) := \mathbb{1}_{\{1/n : n \in \mathbb{N}\}}(x) := \begin{cases} 1 & x = 1/n, n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$ for all $x \in [0,1]$. Show that $f \in \mathcal{R}([0,1])$

   (b) Define $g(x) := \begin{cases} 1/n & x = m/n,\ \gcd(m,n) = 1 \\ 0 & x \notin \mathbb{Q} \end{cases}$ for all $x \in [0,1]$, that is g is the Thomae's function. Show that $g \in \mathcal{R}([0,1])$.

   (c) Show that composition of Riemann integrable functions may not be Riemann integrable.

9. (Very challenging) Let $f : [a, b] \to \mathbb{R}$ be a bounded function.

   (a) Suppose f is continuous except for finitely many points. Show that $f \in \mathcal{R}([a, b])$.

   (b) Suppose f is continuous except for countably many points. Show that $f \in \mathcal{R}([a, b])$.

   (c) Let $A \subset \mathbb{R}$. We say the A is of Lebesgue measure zero if for all $\epsilon > 0$, there exists a countable open interval cover $\{I_i\}_{i=1}^{\infty}$ such that $A \subset \bigcup I_i$ such that $\sum_{i=1}^{\infty} \ell(I_i) < \epsilon$ where $\ell(I_i) := b_i - a_i$ if $I_i = (a_i, b_i)$.

      i. Show that every non-degenerate (non-empty and non-singleton) interval is *not* of Lebesgue measure zero.

      ii. Show that every singleton is of Lebesgue measure zero.

      iii. Show that if $A_1, A_2 \subset A$ is of Lebesgue measure zero, then $A_1 \cup A_2$ is of Lebesgue measure zero.
      (Hint: Consider singletons first for intuition.)

      iv. Show that if (A_n) is a sequence of Lebesgue measure zero sets, then $\bigcup_n A_n$ is of Lebesgue measure zero.
      (Hint: Consider singletons first for intuition.)

      v. Show that if the point of discontinuity of $f : [a, b] \to \mathbb{R}$ is of Lebesgue measure zero, then $f \in \mathcal{R}([a, b])$