Unless otherwise specified, if we write (a, b) or $[a, b]$, it is always the case that $a < b \in \mathbb{R}$.

1 Introduction to Darboux Integration

Definition 1.1. Let $[a, b]$ be a bounded interval. Then

i. We call $P \subset [a, b]$ a **partition** if it is a finite set containing a, b. In particular we can write $P = \{x_i\}_{i=0}^k$ as a finite list where $x_0 = a, x_k = b$ and $x_0 < x_1 \cdots < x_k$. The collection of partitions on $[a, b]$ can be denoted by $\mathcal{P}_{[a,b]}$, or just $\mathcal{P}$ *in this note* for convenience.

ii. For $P, Q \in \mathcal{P}_{[a,b]}$, we say that Q is a **refinement** of P if $P \subset Q$. We also write $P \preceq Q$ if Q refines P. Note that the pair $(\mathcal{P}_{[a,b]}, \preceq)$ forms a partially ordered set.

Definition 1.2. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Let $P := \{x_i\}_{i=0}^k \subset [a, b]$ be a partition. Then

i. We denote $U(f, P) := \sum_{i=1}^k \sup_{x\in[x_{i-1},x_i]} f(x)(x_i - x_{i-1})$ the upper sum of f over P.

ii. We denote $L(f, P) := \sum_{i=1}^k \inf_{x\in[x_{i-1},x_i]} f(x)(x_i - x_{i-1})$ the lower sum of f over P

iii. We denote $\overline{\int_a^b} f := \inf_{P\in\mathcal{P}_{[a,b]}} U(f, P)$ and $\underline{\int_a^b} f := \sup_{P\in\mathcal{P}_{[a,b]}} L(f, P)$ the upper and lower integral of f over $[a, b]$ respectively. It is not hard to see that they are well-defined since f is bounded.

Conceptual Quick Practice

1. Let $(X, \preceq)$ be a partially ordered set. We say that X is a ***directed set*** if every pair of element has an upper bound, that is, for all $x, y \in X$, there exists $z \in X$ such that $z \succeq x$ and $z \succeq y$.

   (a) Show that every totally ordered set is a directed set. Hence $\mathbb{N}$ with the usual order is a directed set.

   (b) Equip $\mathbb{N}$ with the divisibility order , that is, $n \preceq m$ if n is a factor of m. Show that it is a directed partially ordered set that is not totally ordered.

   (c) **(Extremely Important!!!)** Consider a compact interval $[a, b]$. Show that $(\mathcal{P}_{[a,b]}, \preceq)$ with the refinement order is a directed (partially ordered) set.

   Solution. (a). In fact every pair of element in a totally ordered set has a maximum. (b). Upper bounds are given by common multiples. (c). Let $P, Q \subset [a, b]$ be partitions. Then $P \cup Q$ is a partition as it is finite and contains a, b. It is clearly a refinement to both P, Q.

2. Following Q1, we are defining more general notions for convergence. Let I be a directed set. Then any function $f : I \to \mathbb{R}$ is said to be a ***net*** over I. By convention, We write $f_i := f(i)$ for all $i \in I$ and denote the net as $f = (f_i)_{i\in I}$. A net is increasing (or decreasing) if it shares the same property as a function.

   (a) Show that every sequence can be regarded as a net over $\mathbb{N}$.

   (b) Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Consider $U_P := U(f, P)$ for all partition $P \in (\mathcal{P}_{[a,b]}, \preceq)$. Show that $(U_P)_{P\in\mathcal{P}}$ is a decreasing net.

   (c) Following (b), define $L_P := L(f, P)$ for all $P \in \mathcal{P}$. Show that $(L_P)_{P\in\mathcal{P}}$ is an increasing net.

   Solution. (a). It follows from the fact that $\mathbb{N}$ is a directed set. (b). It is equivalent to show that $P \subset Q$ would imply $U_P \geq U_Q$ where P, Q are partitions. Consider first the case $Q = P \cup \{x\}$ where $x \notin P$. It clearly follows that $U(f, Q) \leq U(f, P)$. The general result follows from the finiteness of Q. (c) is similar to (b).

3. Let I be a directed set. Consider $x := (x_i)_{i\in I}$ a net of real numbers. Then we say that (x_i) converges to some real numbers x if for all $\epsilon > 0$, there exists $\Lambda \in I$ such that for all $i \succeq \Lambda$, we have $|x_i - x| < \epsilon$.

   (a) Let $(x_i)_{i\in I}$ be a net. Suppose x, y are limits of $(x_i)_{i\in I}$. Show that $x = y$.

   (b) Part (a) showed that limits of nets are unique if they exist. We can write $\lim_i x_i := x$ if x is the limit of the net $(x_i)_{i\in I}$. Suppose $(x_i), (y_i)$ are *convergent nets* over I. Show that

      i. $\lim_i(x_i + y_i) = \lim_i x_i + \lim_i y_i$

      ii. $\lim_i x_i \le \lim_i y_i$ if there exists $\Lambda \in I$ such that $x_i \le y_i$ for all $i \succeq \Lambda$

   (c) Is it true that a converging net is always bounded (defined by considering a net as a function)? *(Hint: Consider the index set to be the set of all real numbers.)*

   Solution. (a). Let $\epsilon > 0$. Then there exists Λ_1, Λ_2 such that we have $|x_i - x| < \epsilon$ if $i \succeq \Lambda_1$ and $|x_i - y| < \epsilon$ if $i \succeq \Lambda_2$. By directedness, there exists $\Lambda \in I$ such that $\Lambda \succeq \Lambda_1, \Lambda_2$. Hence we have $|x - y| \le |x_i - x| + |x_i - y| < 2\epsilon$ by considering some $i \succeq \Lambda$. It follows that $x = y$ as $\epsilon \to 0$.
   (b.i). Note that $|x_i + y_i - x - y| \le |x_i - x| + |y_i - y|$ where $x := \lim x_i$ and $y := \lim_i y_i$ for all $i \in I$. Let $\epsilon > 0$. Then the proof proceeds as the sequential case. (b.ii). By (i), it suffices to consider the case $x_i = 0$ for all $i \in I$. Suppose $y := \lim y_i < 0$. Then $-y > -y/2 > 0$. It follows that there exists $\Lambda' \in I$ such that $i \succeq \Lambda'$ would imply $|y_i - y| < -y/2$. This give a contradiction by considering some $i \succeq \Lambda$ and Λ'. (c). Consider the function $f(t) := 1/t$ on $(0, \infty)$. Then it is unbounded but converges as a net. In fact $\lim_t f(t) = 0$.

4. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Show that $\lim_P U(f, P) = \overline{\int_a^b} f := \inf_{P\in\mathcal{P}} U(f, P)$ and $\lim_P L(f, P) = \underline{\int_a^b} f := \sup_{P\in\mathcal{P}} L(f, P)$ where convergence in nets is used.

   Solution. Use the fact that $U(f, P)$ and $L(f, P)$ are decreasing and increasing nets respectively.

More Quick Practice

1. Let $A \subset \mathbb{R}$. We define $\mathbb{1}_A(x) := \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$ for all $x \in \mathbb{R}$ to be the ***indicator function*** of A.

   (a) Let $A \subset [0, 1]$ be a singleton. Show that $\overline{\int_0^1} \mathbb{1}_A = \underline{\int_0^1} \mathbb{1}_A = 0$.

   (b) Let $A \subset [0, 1]$ be a finite set. Show that $\overline{\int_0^1} \mathbb{1}_A = \underline{\int_0^1} \mathbb{1}_A = 0$.

   (c) Let $A \subset [0, 1]$ be a countable set. Is it always true that $\overline{\int_0^1} \mathbb{1}_A = \underline{\int_0^1} \mathbb{1}_A = 0$?

   Solution. (a) follows from (b). (b). Write $A = \{x_i\}_{i=1}^k$. Then $d := \min_{i\neq j} |x_i - x_j| > 0$. Let $\epsilon > 0$. Consider the partition $P = \{0, 1\} \cup \{x_i \pm \epsilon d/4k\} \cap [0, 1]$. Then it is not hard to see that $0 \le \overline{\int_0^1} 1_{\mathbb{A}} \le U(\mathbb{1}_A, P) = \sum_{i=1}^k 2\epsilon d/4k = \epsilon d/2 < \epsilon$. The result follows as $\epsilon \to 0$. (c). No. Consider $A = \mathbb{Q} \cap [0, 1]$.

2. Let $f, g : [a, b] \to \mathbb{R}$ be bounded functions. Suppose $f = g$ except for a finitely many points on $[a, b]$. Show that $\overline{\int_a^b} f = \overline{\int_a^b} g$. Is it true that $\underline{\int_a^b} f = \underline{\int_a^b} g$?

   Solution. The answer to the question is true. We prove only for the case of upper integrals. Write $L := \overline{\int_a^b} f$. Let $\epsilon > 0$. Then there exists a partition P such that $U(f, P) - L < \epsilon$. Consider a partition Q similar to that in Q1b such that $|U(f, R) - U(g, R)| < \epsilon$ for all refinement R of Q. It follows that $|U(g, T) - L| < 2\epsilon$ for all T refining P, Q. Hence, $\lim_T U(g, T) = \overline{\int_a^b} g = L = \overline{\int_a^b} f$.

3. Let $f, g : [a, b] \to \mathbb{R}$ be bounded functions.

   (a) Show that $U(f + g, P) \le U(f, P) + U(g, P)$ for all partition $P \in \mathcal{P}_{[a,b]}$.

   (b) Hence, show that $\overline{\int_a^b}(f + g) \le \overline{\int_a^b} f + \overline{\int_a^b} g$.

   (c) Find examples for both the equality and strict inequality case in (b).

   (d) Is it true that $\underline{\int_a^b}(f + g) \le \underline{\int_a^b} f + \underline{\int_a^b} g$? If it is false, give a similar inequality that should hold.

   Solution. (a) is easy. (b) follows by taking limits of nets in (a) with the help of Q3 in Conceptual Quick Practice. (c). Consider $f = \mathbb{1}_{\mathbb{Q}}$ and $g = -f$ or $g = f$. (d). No. We should have $\underline{\int_a^b}(f + g) \ge \underline{\int_a^b} f + \underline{\int_a^b} g$

4. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Show that

   (a) For all $\lambda \geq 0$, we have $\overline{\int_a^b} \lambda f = \lambda \overline{\int_a^b}$; for all $\lambda < 0$, we have $\overline{\int_a^b} \lambda f = \lambda \underline{\int_a^b} f$

   (b) The function defined by $f \mapsto \overline{\int_a^b} f$ is a convex function over the space of bounded functions, that is, for all $\lambda \in [0, 1]$ and $f, g : [a, b] \to \mathbb{R}$ bounded, we have $\overline{\int_a^b} \lambda f + (1 - \lambda)g \leq \lambda \overline{\int_a^b} f + (1 - \lambda) \overline{\int_a^b} g$

   Solution. (a). Consider first the upper or lower sums. (b). This follow from 3(b) and 4(a).

5. Let $f : [a, b] \to \mathbb{R}$ be a bounded function.

   (a) Let $g : [a, b] \to \mathbb{R}$ be bounded such that $g \geq f$ on $[a, b]$ pointwise. Show that $\overline{\int_a^b} g \geq \overline{\int_a^b} f$ and $\underline{\int_a^b} g \geq \underline{\int_a^b} f$.

   (b) Show that we have $\left| \overline{\int_a^b} f \right| \leq \overline{\int_a^b} |f|$. Is it true that we have $\left| \underline{\int_a^b} f \right| \leq \underline{\int_a^b} |f|$?

   Solution. (a). Consider first the upper or lower sums. Then take limits of suitable nets. (b). As $-|f| \leq f \leq |f|$, we have $-\overline{\int_a^b} f \leq -\underline{\int_a^b} |f| \overset{(*)}{=} \overline{\int_a^b} -|f| \leq \overline{\int_a^b} f \leq \overline{\int_a^b} |f|$, in which we have used scalar multiplication property of upper and lower integrals at $(*)$. The function $f = -\mathbb{1}_{\mathbb{Q}}$ gives a counter-example to the lower integral case.[1]

6. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Suppose $f \geq 0$ on $[a, b]$.

   (a) Suppose f is continuous. Show that $f \equiv 0$ on $[a, b]$ if and only if $\underline{\int_a^b} f = 0$

   (b) Can the continuity assumption in (a) be dropped? Provide suitable examples whenver necessary.

   Solution. (a). Only $(\Leftarrow)$ is non-trivial. Suppose not. Then $f(c) > 0$ for some $c \in [a, b]$. Hence $f(x) > f(c)/2 > 0$ for all $x \in B_r(c) \subset [a, b]$ for some $r > 0$. It is then not hard to see that $\underline{\int_a^b} f \geq \underline{\int_{x-r}^{x+r}} f \geq 2rf(c)/2 > 0$. (b). No. Consider $f = \mathbb{1}_{\mathbb{Q}}$.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded function. We define upper and lower integrals of f over a compact interval $[a, b]$ by considering the restriction $f|_{[a,b]}$. Show that $\overline{\int_a^b} f + \overline{\int_b^c} f = \overline{\int_a^c} f$ for all $a < b < c$.

   Solution. Let P, Q be partitions of $[a, b]$ and $[b, c]$ respectively. Then $P \cup Q$ is a partition of $[a, c]$ and we clearly have $U(f, P) + U(f, Q) = U(f, P \cup Q) \geq \overline{\int_a^c} f$. It follows that $\overline{\int_a^c} f \leq \overline{\int_a^b} f + \overline{\int_b^c} f$ by taking limits through P, Q. For the other side, let R be a partition of $[a, c]$. Then clearly $R \cup \{b\}$ is a refinement that can be broken into partitions on $[a, b]$ and $[b, c]$. It follows that $U(f, R) \geq U(f, R \cup \{b\}) \geq \overline{\int_a^b} f + \overline{\int_b^c} f$. The result follows by taking limits on R. (cf. Lecture note: the proof here is similar to the ordinary integral)

8. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Define the function $F(x) := \overline{\int_a^x} f$ for all $x \in [a, b]$. Show that F is Lipschitz continuous on $[a, b]$

   Solution. Let $x < y$. Note $|F(x) - F(y)| = \left| \overline{\int_x^y} f \right| \leq \overline{\int_x^y} |f| \leq |x - y| \sup_{t \in [a,b]} |f(t)|$ by Q7 and Q5.

9. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that f' is bounded. Show that for all $x < y \in \mathbb{R}$, we have

$$\underline{\int_x^y} f' \leq f(y) - f(x) \leq \overline{\int_x^y} f'$$

   Solution. Let $P = \{x_i\}_{i=0}^k$ be a partition of $[x, y]$. Then $f(y) - f(x) = \sum_{i=1}^k f(x_i) - f(x_{i-1}) = \sum_{i=1}^k f'(\xi_i)(x_i - x_{i-1})$ by MVT where $\xi_i \in (x_{i-1}, x_i)$. It is then clear that $L(f', P) \leq f(y) - f(x) \leq U(f', P)$. The result follows by taking limit for P as P is arbitrary.

[1] Many thanks to Matthew Liu who pointed out this mistake and provided the counter-example.

10. We say that $f : [0,1] \to \mathbb{R}$ is a ***step function*** over $[0,1]$ if it is a linear combination of indicators of disjoint intervals, that is, there exists $\{I_i\}_{i=1}^k$ where $I_i \subset [0,1]$ are disjoint intervals (of any form) and a list of real numbers $\{a_i\}_{i=1}^k$ such that $f = \sum_{i=1}^k a_i \mathbb{1}_{I_i}$. Let $P := \{x_i\}_{i=0}^k$ be partition of $[0,1]$. Let $(a_i)_{i=1}^k$ be a sequence of real numbers. Define the step function $f := \sum_{i=1}^k a_i \mathbb{1}_{[x_{i-1},x_i)}$. Show that

$$\overline{\int_0^1} f = \underline{\int_0^1} f = \sum_{i=1}^k a_i(x_i - x_{i-1})$$

Solution. Simplify the integrals by partitioning it into intervals with respect to P using Q7. Then use Q2 to integrate only constant functions. (cf. HW 4 Solutions)