1 Introduction to Darboux Integration

Definition 1.1. Let $[a, b]$ be a bounded interval. Then

i. We call $P \subset [a, b]$ a **partition** if it is a finite set containing a, b. In particular we can write $P = \{x_i\}_{i=0}^k$ as a finite list where $x_0 = a, x_k = b$ and $x_0 < x_1 \cdots < x_k$. The collection of partitions on $[a, b]$ can be denoted by $\mathcal{P}_{[a,b]}$, or just $\mathcal{P}$ *in this note* for convenience.

ii. For $P, Q \in \mathcal{P}_{[a,b]}$, we say that Q is a **refinement** of P if $P \subset Q$. We also write $P \preceq Q$ if Q refines P. Note that the pair $(\mathcal{P}_{[a,b]}, \preceq)$ forms a partially ordered set.

Definition 1.2. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Let $P := \{x_i\}_{i=0}^k \subset [a, b]$ be a partition. Then

i. We denote $U(f, P) := \sum_{i=1}^k \sup_{x \in [x_{i-1}, x_i]} f(x)(x_i - x_{i-1})$ the upper sum of f over P.

ii. We denote $L(f, P) := \sum_{i=1}^k \inf_{x \in [x_{i-1}, x_i]} f(x)(x_i - x_{i-1})$ the lower sum of f over P

iii. We denote $\overline{\int_a^b} f := \inf_{P \in \mathcal{P}_{[a,b]}} U(f, P)$ and $\underline{\int_a^b} f := \sup_{P \in \mathcal{P}_{[a,b]}} L(f, P)$ the upper and lower integral of f over $[a, b]$ respectively. It is not hard to see that they are well-defined since f is bounded.

Conceptual Quick Practice

1. Let $(X, \preceq)$ be a partially ordered set. We say that X is a ***directed set*** if every pair of element has an upper bound, that is, for all $x, y \in X$, there exists $z \in X$ such that $z \succeq x$ and $z \succeq y$.

   (a) Show that every totally ordered set is a directed set. Hence $\mathbb{N}$ with the usual order is a directed set.

   (b) Equip $\mathbb{N}$ with the divisibility order , that is, $n \preceq m$ if n is a factor of m. Show that $\mathbb{N}$ is a directed partially ordered set that is not totally ordered.

   (c) **(Extremely Important!!!)** Consider a compact interval $[a, b]$. Show that $(\mathcal{P}_{[a,b]}, \preceq)$ with the refinement order is a directed (partially ordered) set.

2. Following Q1, we are defining more general notions for convergence. Let I be a directed set. Then any function $f : I \to \mathbb{R}$ is said to be a ***net*** over I. By convention, We write $f_i := f(i)$ for all $i \in I$ and denote the net as $f = (f_i)_{i \in I}$. A net is increasing (or decreasing) if it shares the same property as a function.

   (a) Show that every sequence can be regarded as a net over $\mathbb{N}$.

   (b) Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Consider $U_P := U(f, P)$ for all partition $P \in (\mathcal{P}_{[a,b]}, \preceq)$. Show that $(U_P)_{P \in \mathcal{P}}$ is a decreasing net.

   (c) Following (b), define $L_P := L(f, P)$ for all $P \in \mathcal{P}$. Show that $(L_P)_{P \in \mathcal{P}}$ is an increasing net.

3. Let I be a directed set. Consider $x := (x_i)_{i\in I}$ a net of real numbers. Then we say that (x_i) converges to some real numbers x if for all $\epsilon > 0$, there exists $\Lambda \in I$ such that for all $i \succeq \Lambda$, we have $|x_i - x| < \epsilon$.
   (a) Let $(x_i)_{i\in I}$ be a net. Suppose x, y are limits of $(x_i)_{i\in I}$. Show that $x = y$.
   (b) Part (a) showed that limits of nets are unique if they exist. We can write $\lim_i x_i := x$ if x is the limit of the net $(x_i)_{i\in I}$. Suppose $(x_i), (y_i)$ are *convergent nets* over I. Show that
   - $\lim_i (x_i + y_i) = \lim_i x_i + \lim_i y_i$
   - $\lim_i x_i \leq \lim_i y_i$ if there exists $\Lambda \in I$ such that $x_i \leq y_i$ for all $i \succeq \Lambda$

   (c) Is it true that a converging net is always bounded (defined by considering a net as a function)? *(Hint: Consider the index set to be the set of all real numbers.)*

4. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Show that $\lim_P U(f, P) = \overline{\int_a^b} f := \inf_{P\in\mathcal{P}} U(f, P)$ and $\lim_P L(f, P) = \underline{\int_a^b} f := \sup_{P\in\mathcal{P}} L(f, P)$ where convergence in nets is used.

More Quick Practice

1. Let $A \subset \mathbb{R}$. We define $\mathbb{1}_A(x) := \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$ for all $x \in \mathbb{R}$ to be the ***indicator function*** of A.
   (a) Let $A \subset [0, 1]$ be a singleton. Show that $\overline{\int_0^1} \mathbb{1}_A = \underline{\int_0^1} \mathbb{1}_A = 0$.
   (b) Let $A \subset [0, 1]$ be a finite set. Show that $\overline{\int_0^1} \mathbb{1}_A = \underline{\int_0^1} \mathbb{1}_A = 0$.
   (c) Let $A \subset [0, 1]$ be a countable set. Is it always true that $\overline{\int_0^1} \mathbb{1}_A = \underline{\int_0^1} \mathbb{1}_A = 0$?

2. Let $f, g : [a, b] \to \mathbb{R}$ be bounded functions. Suppose $f = g$ except for a finitely many points on $[a, b]$. Show that $\overline{\int_a^b} f = \overline{\int_a^b} g$. Is it true that $\underline{\int_a^b} f = \underline{\int_a^b} g$?

3. Let $f, g : [a, b] \to \mathbb{R}$ be bounded functions.
   (a) Show that $U(f + g, P) \leq U(f, P) + U(g, P)$ for all partition $P \in \mathcal{P}_{[a,b]}$.
   (b) Hence, show that $\overline{\int_a^b}(f + g) \leq \overline{\int_a^b} f + \overline{\int_a^b} g$.
   (c) Find examples for both the equality and strict inequality case in (b).
   (d) Is it true that $\underline{\int_a^b}(f + g) \leq \underline{\int_a^b} f + \underline{\int_a^b} g$? If it is false, give a similar inequality that should hold.

4. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Show that
   (a) For all $\lambda \geq 0$, we have $\overline{\int_a^b} \lambda f = \lambda \overline{\int_a^b}$; for all $\lambda < 0$, we have $\overline{\int_a^b} \lambda f = \lambda \underline{\int_a^b} f$
   (b) The function defined by $f \mapsto \overline{\int_a^b} f$ is a convex function over the space of bounded functions, that is, for all $\lambda \in [0, 1]$ and $f, g : [a, b] \to \mathbb{R}$ bounded, we have $\overline{\int_a^b} \lambda f + (1 - \lambda)g \leq \lambda \overline{\int_a^b} f + (1 - \lambda) \overline{\int_a^b} g$

5. Let $f : [a, b] \to \mathbb{R}$ be a bounded function.
   (a) Let $g : [a, b] \to \mathbb{R}$ be bounded such that $g \geq f$ on $[a, b]$ pointwise. Show that $\overline{\int_a^b} g \geq \overline{\int_a^b} f$ and $\underline{\int_a^b} g \geq \underline{\int_a^b} f$.
   (b) Show that we have $\left| \overline{\int_a^b} f \right| \leq \overline{\int_a^b} |f|$. Is it true that we have $\left| \underline{\int_a^b} f \right| \leq \underline{\int_a^b} |f|$?

6. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Suppose $f \geq 0$ on $[a, b]$.
   (a) Suppose f is continuous. Show that $f \equiv 0$ on $[a, b]$ if and only if $\underline{\int_a^b} f = 0$
   (b) Can the continuity assumption in (a) be dropped? Provide suitable examples whenver necessary.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded function. We define upper and lower integrals of f over a compact interval $[a, b]$ by considering the restriction $f |_{[a,b]}$. Show that $\overline{\int_a^b} f + \overline{\int_b^c} f = \overline{\int_a^c} f$ for all $a < b < c$.

8. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Define the function $F(x) := \overline{\int_a^x} f$ for all $x \in [a, b]$. Show that F is Lipschitz continuous on $[a, b]$

9. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that f' is bounded. Show that for all $x < y \in \mathbb{R}$, we have

$$\underline{\int_x^y} f' \leq f(x) - f(y) \leq \overline{\int_x^y} f'$$

10. We say that $f : [0, 1] \to \mathbb{R}$ is a ***step function*** over $[0, 1]$ if it is a linear combination of indicators of disjoint intervals, that is, there exists $\{I_i\}_{i=1}^k$ where $I_i \subset [0, 1]$ are disjoint intervals (of any form) and a list of real numbers $\{a_i\}_{i=1}^k$ such that $f = \sum_{i=1}^k a_i \mathbb{1}_{I_i}$. Let $P := \{x_i\}_{i=0}^k$ be partition of $[0, 1]$. Let $(a_i)_{i=1}^k$ be a sequence of real numbers. Define the step function $f := \sum_{i=1}^k a_i \mathbb{1}_{[x_{i-1}, x_i)}$. Show that

$$\overline{\int_0^1} f = \underline{\int_0^1} f = \sum_{i=1}^k a_i (x_i - x_{i-1})$$