Unless otherwise specified, we always use I to denote an open interval. If we write (a, b) or $[a, b]$, it is always the case that $a < b \in \mathbb{R}$.

1 Mean Value Theorem

Theorem 1.1 (Mean Value Theorem). *Let $f : [a, b] \to \mathbb{R}$ be continuous. Suppose f is differentiable on (a, b). Then there exists $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$. In particular, if $f : I \to \mathbb{R}$ is differentiable, then for all $x < y \in I$, there exists $\xi \in (x, y)$ such that $f(y) - f(x) = f'(\xi)(y - x)$.*

Proposition 1.2 (Derivative of local extrema). *Let $f : I \to \mathbb{R}$. Suppose f is differentiable and f is a local maximum at c, that is, $f(c) \geq f(x)$ for all $x \in B_r(c)$ for some $r > 0$. Then $f'(c) = 0$.*

Quick Practice

1. Show that if $f : I \to \mathbb{R}$ is differentiable such that $f' \equiv 0$ on I, then f is a constant on I.

   Solution. Let $x < y \in I$. Then $f(x) - f(y) = f'(\xi)(x - y) = 0$ for some $\xi \in (x, y)$. It follows that $f(x) = f(y)$ for all $x < y$. In particular f is constant on I.

2. Let $f : I \to \mathbb{R}$ be a differentiable function.

   (a) Suppose $f' \geq 0$ on I. Show that f is an increasing function, that is, $f(x) \geq f(y)$ for all $x \geq y \in I$.

   (b) Suppose $f' > 0$ on I. Is it true that f' is a strictly increasing function, that is, $f(x) > f(y)$ for all $x > y \in I$? Give counterexamples whenever necessary.

   Solution. Both parts are similar. We shall do part 2(b) only: it is true. Let $x > y \in I$. Then $f(x) - f(y) = f'(\xi)(x - y)$ for $\xi \in (y, x)$ by MVT. Since $f'(\xi) > 0$ and $x - y > 0$, we have $f(x) - f(y) > 0$ and so $f(x) > f(y)$

   Remark. The converse of (a) is correct while the converse of (b) is incorrect. For the latter, consider $f(x) := x^3$ for $x \in \mathbb{R}$

3. Let $f(x) := \sin(x)$ for all $x \in \mathbb{R}$. Show that f is a 1-Lipschitz function on $\mathbb{R}$, that is, $|f(x) - f(y)| \leq |x - y|$ for all $x, y \in \mathbb{R}$.

   Solution. Note that $f'(x) = \cos(x)$ for all $x \in \mathbb{R}$. Let $x < y \in \mathbb{R}$, by MVT, we have $|f(x) - f(y)| = |\cos(\xi)||x - y| \leq |x - y|$.

4. Let $f : I \to \mathbb{R}$ for all $x \in \mathbb{R}$. Suppose f is differentiable and f' is bounded on I.

   (a) Show that f is Lipschitz on I.

   (b) Show that if $|f'| \leq M$ on I for $M > 0$. Then f is a $M-$Lipschitz function.

   (c) (Repeat) Suppose f is differentiable. Is it true that f is Lipschitz if and only if f' is bounded. Prove your assertion.

   (d) (Repeat) Give an example such that f is differentiable but is not Lipschitz.

   Solution. (a) and (b). Let $x, y \in I$. By MVT, $|f(x) - f(y)| = |f'(\xi)||x - y| \leq M|x - y|$ where $|f'| \leq M$ on I. It follows that f is $M-$Lipschitz.
   (c). Only $(\Rightarrow)$ has not been proved: fix $c \in I$. Then for all $x \neq c \in I$, we have $|f(x) - f(c)| \leq L|x - c|$ where L is the Lipschitz-constant. It follows that $|f'(c)| = \lim_{x \to c} \left|\frac{f(x)-f(c)}{x-c}\right| \leq L$. Since c is arbitrary, we have $|f'| \leq L$.
   (d). Consider any f with f' unbounded.

5. Define $f(x) := e^x$ for all $x \in \mathbb{R}$. Show that f is not Lipschitz on $\mathbb{R}$. Nevertheless, f is Lipschitz for all on $(-\infty, t)$ for all $t > 0$.

   Solution. Note that $f'(x) = e^x$ for all $x \in \mathbb{R}$. f is not Lipschitz on $\mathbb{R}$ because f' is unbounded on $\mathbb{R}$. Nevertheless, fix $t > 0$ then f is Lipschitz on $(-\infty, t)$ because f' is bounded with $|f'| \leq e^t$.

2 Taylor's Theorem

Theorem 2.1 (Taylor's Theorem). *Let $f : I \to \mathbb{R}$ be $(n+1)-$times differentiable for $n \geq 0$. Then for all $x < y \in I$, there exists $\xi \in (x, y)$ such that*

$$f(y) - f(x) = \sum_{i=1}^{n} \frac{f^{(n)}(x)}{n!}(y-x)^n + \frac{f^{n+1}(\xi)}{(n+1)!}(y-x)^{(n+1)}$$

Quick Practice

1. (Ex6.4 Q4) Let $x > 0$. Show that $1 + x/2 - x^2/8 \leq \sqrt{1+x} \leq 1 + x/2$

   Solution. Homework candidate: skip.

2. (Ex 6.4 Q13) Calculate e, correct to 7 decimal places.

   Solution. Consider $f(t) := e^t$ with $x = 1$ and $y = 2$ (using the notations above). Use the nth Taylor's polynomial to approximate where $\frac{e^2}{(n+1)!} \leq 5 \times 10^{-8}$.

3. Let $f : I \to \mathbb{R}$. Suppose f is twice differentiable on I. Show that $f^{(2)} \geq 0$ on I if and only if f is convex, that is, $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$ for all $x, y \in I$ and $t \in [0, 1]$

   Solution. See Lecture Notes. Hint: One technique for dealing with convexity is to work with $z \in [x, y]$ instead of $tx + (1-t)y$ where $t \in [0, 1]$. Note that there is a bijection between $z \in [x, y]$ and $t \in [0, 1]$ by $t \mapsto tx + (1-t)y \in [x, y]$.

3 Extra Exercises

1. Let $f : I \to \mathbb{R}$ be differentiable.

   (a) Show that if f' has a right-limit at $c \in I$, then f' is right-continuous at c, that is, if $\lim_{x\to c^+} f'(x) \in \mathbb{R}$ then $f'(c) = \lim_{x\to c^+} f'(x)$.

   (b) Suppose f' is increasing. Show that f' is continuous.

   Solution. (a). Let $x > c \in I$. Then $f(x) - f(c) = f'(\xi(x))(x - c)$ where $\xi(c) \in (c, x)$. Then $f'(\xi(x)) = \frac{f(x)-f(c)}{x-c}$. Now we consider $x \to c^+$ on both side. For the right expression, $\lim_{x\to c^+} \frac{f(x)-f(c)}{x-c} = f'(c)$ as f is differentiable at c. For ther left expression, we have to show that $\lim_{x\to c^+} f'(\xi(x)) = \lim_{x\to c^+} f(x)$. Write $L := \lim_{x\to c^+} f(x)$. Let $\epsilon > 0$. Then there exists $\delta > 0$ such that $x - c < \delta$ would imply $|f(x) - L| < \epsilon$. Note that if $x - c < \delta$ then $\xi(x) - c < x - c < \delta$. In particular, we have $|f(\xi(x)) - L| < \epsilon$. It follows that $\lim_{x\to c^+} f'(\xi(x)) = L$.
   (b). First note that clearly part (a) is still true if we replace "right" by "left". Note that an increasing function on I has both left and right limits at all points on I by considering supremums and infimums. It follows that if f' is increasing then f' is both left and right continuous at all points. In particular, f is continuous at all points.

2. **(Modified)** Suppose $f : I \to \mathbb{R}$. We say that f is locally strictly increasing at $c \in I$ if there exists $r > 0$ such that f is strictly increasing on $B_r(c) \subset I$. Suppose f is differentiable such that f' is continuous, i.e. $f \in C^1(I)$.

   (a) Show that if $f'(c) > 0$ then f is locally strictly increasing at c.

   (b) Is the converse of the above true? Prove your assertion.

   (c) Show that if f is locally increasing at c, that is, f is increasing on $B_r(c)$ for some $r > 0$ (with partial inequality), then $f'(c) \geq 0$.

   (d) Suppose $f'(c) \geq 0$. Is it true that f is locally increasing at c? Prove your assertion.

   (e) Suppose now f is differentiable but f' may not be continuous. Do statements in (a) - (d) still hold or not?

   Solution. (a). Note $f' > 0$ on $B_r(c)$ for some $r > 0$ by continuity. Hence the result follows from MVT on $B_r(c)$. (b) is not by considering $f(x) := x^3$ as in the first page. (c). It is easy to see that $\frac{f(x)-f(c)}{x-c} \geq 0$ on $x \in B_r(c) \backslash \{c\}$. The result follows by taking limit. (d). It is not true. Consider $f(x) = x^2$ then $f'(0) = 0$ but f is not locally increasing at 0.
   (e). The same proof and examples apply for (b), (c), (d). For (a), the function $g(x) := x + 2x^2 \sin(1/x)$ for $x \neq 0$ and $g(0) = 0$ gives a counter example. (cf. textbook P. 179 Q10)

3. Let $f : I \to \mathbb{R}$ be a function. We say that f has the *Intermediate Value Property* if for all $x, y \in I$ such that $f(x) < f(y)$ and for all $t \in [f(x), f(y)]$, there exists $z \in [x, y]$ or $z \in [y, x]$ such that $f(z) = t$.

   (a) Show that if f is a continuous function, then f has the Intermediate Value Property.

   (b) Suppose f is differentiable on I. Suppose further that $x < y \in I$ such that $f'(x) < 0 < f'(y)$. Show that there exists $z \in (x, y)$ such that $f'(z) = 0$.

   (c) Suppose f is differentiable. Show that f' has the Intermediate Value Property.

   (d) Find a non-continuous function that has the Intermediate Value Property.

   Solution. Read the Darboux's Theorem in textbook for details. It is interesting to note that there exists a function that is nowhere continuous but satisfy the Intermediate Value Property. See the Wikipage of the Conway Base 13 function.

4. Let $f : [0, \infty)$ be a function. Let $t > 0$. We call a finite list of points $\{a_i\}_{i=-0}^{n}$ a partition of $[0, t]$ if $0 = a_0 < a_1 < \cdots < a_n := t$. We define

$$V_f(t) := V([0,t]) := \sup\{\sum_{i=1}^{n} |f(a_i) - f(a_{i-1})| : \{a_i\}_{i=0}^{n} \textit{ is a partition of } [0,t]\}$$

   Note that $V_f(t) \in [0, \infty]$ can take value $+\infty$. We call f to be of finite variation if $V_f(t) < \infty$ for all $t \geq 0$.

   (a) Suppose $f \in \mathcal{C}^1([0, \infty))$. Show that f is of finite variation.

   (b) Suppose $f \in \mathcal{C}^1([0, \infty))$. Show that f is the sum of two monotone functions. (*Hint: Show that $t \mapsto V_f(t)$ is an increasing function on* $[0, \infty)$).

   Solution. (a). Use the fact that f' is continuous (by definition) on $[0, t]$ for $t > 0$ and so f' is bounded on $[0, t]$. Therefore f is Lipschitz on $[0, t]$. Then one can proceed by using the triangle inequality.
   (b). The decomposition $f(t) = V_f(t) + (f - V_f)(t)$ is the required sum. Note that $V_f(t)$ and $(f - V_f)(t) := f(t) - V_f(t)$ are both monotone.

5. Let $f : I \to \mathbb{R}$ be a function.

   (a) Show that f is convex if and only if for all finite list $\{\lambda_i\}_{i=1}^{n} \subset [0, 1]$ and $\{x_i\}_{i=1}^{n} \subset I$ such that $\sum_{i=1}^{n} \lambda_i = 1$, we have

$$f(\sum_{i=1}^{n} \lambda_i x_i) \leq \sum_{i=1}^{n} \lambda_i f(x_i)$$

   (b) Show that for all $n \in \mathbb{N}$ and $\{x_i\}_{i=1}^{n} \subset \mathbb{R}_{>0}$. We have $(x_1 \cdots x_n)^{1/n} \leq \dfrac{x_1 + \cdots + x_n}{n}$

   Solution. (a). By a standard induction argument.
   (b). Note that e^x is convex by the second derivative test. Write $x_i = e^{\log(x_i)}$ and $\lambda_i := 1/n$ for all $i = 1, \cdots, n$ and $n \in \mathbb{N}$. Then apply part (a) on the convexity of e^x.