Unless otherwise specified, we always use I to denote an open interval. If we write (a, b) or $[a, b]$, it is always the case that $a < b \in \mathbb{R}$.

1 Mean Value Theorem

Theorem 1.1 (Mean Value Theorem). *Let $f : [a, b] \to \mathbb{R}$ be continuous. Suppose f is differentiable on (a, b). Then there exists $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$. In particular, if $f : I \to \mathbb{R}$ is differentiable, then for all $x < y \in I$, there exists $\xi \in (x, y)$ such that $f(y) - f(x) = f'(\xi)(y - x)$.*

Proposition 1.2 (Derivative of local extrema). *Let $f : I \to \mathbb{R}$. Suppose f is differentiable and f is a local maximum at c, that is, $f(c) \geq f(x)$ for all $x \in B_r(c)$ for some $r > 0$. Then $f'(c) = 0$.*

Quick Practice

1. Show that if $f : I \to \mathbb{R}$ is differentiable such that $f' \equiv 0$ on I, then f is a constant on I.

2. Let $f : I \to \mathbb{R}$ be a differentiable function.
   (a) Suppose $f' \geq 0$ on I. Show that f is an increasing function, that is, $f(x) \geq f(y)$ for all $x \geq y \in I$.
   (b) Suppose $f' > 0$ on I. Is it true that f' is a strictly increasing function, that is, $f(x) > f(y)$ for all $x > y \in I$? Give counterexamples whenever necessary.

3. Let $f(x) := \sin(x)$ for all $x \in \mathbb{R}$. Show that f is a 1-Lipschitz function on $\mathbb{R}$, that is, $|f(x) - f(y)| \leq |x - y|$ for all $x, y \in \mathbb{R}$.

4. Let $f : I \to \mathbb{R}$ for all $x \in \mathbb{R}$. Suppose f is differentiable and f' is bounded on I.
   (a) Show that f is Lipschitz on I.
   (b) Show that if $|f'| \leq M$ on I for $M > 0$. Then f is a $M-$Lipschitz function.
   (c) (Repeat) Suppose f is differentiable. Is it true that f is Lipschitz if and only if f' is bounded. Prove your assertion.
   (d) (Repeat) Give an example such that f is differentiable but is not Lipschitz.

5. Define $f(x) := e^x$ for all $x \in \mathbb{R}$. Show that f is not Lipschitz on $\mathbb{R}$. Nevertheless, f is Lipschitz for all on $(-\infty, t)$ for all $t > 0$.

2 Taylor's Theorem

Theorem 2.1 (Taylor's Theorem)**.** *Let $f : I \to \mathbb{R}$ be $(n+1)-$times differentiable for $n \geq 0$. Then for all $x < y \in I$, there exists $\xi \in (x, y)$ such that*

$$f(y) - f(x) = \sum_{i=1}^{n} \frac{f^{(n)}(x)}{n!}(y-x)^n + \frac{f^{n+1}(\xi)}{(n+1)!}(y-x)^{(n+1)}$$

Quick Practice

1. (Ex6.4 Q4) Let $x > 0$. Show that $1 + x/2 - x^2/8 \leq \sqrt{1+x} \leq 1 + x/2$

2. (Ex 6.4 Q13) Calculate e, correct to 7 decimal places.

3. Let $f : I \to \mathbb{R}$. Suppose f is twice differentiable on I. Show that $f^{(2)} \geq 0$ on I if and only if f is convex, that is, $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$ for all $x, y \in I$ and $t \in [0, 1]$

3 Extra Exercises

1. Let $f : I \to \mathbb{R}$ be differentiable.
   (a) Show that if f' has a right-limit at $c \in I$, then f' is right-continuous at c, that is, if $\lim_{x \to c^+} f'(x) \in \mathbb{R}$ then $f'(c) = \lim_{x \to c^+} f'(x)$.
   (b) Suppose f' is increasing. Show that f' is continuous.

2. **(Modified)** Suppose $f : I \to \mathbb{R}$. We say that f is locally strictly increasing at $c \in I$ if there exists $r > 0$ such that f is strictly increasing on $B_r(c) \subset I$. Suppose f is differentiable such that f' is continuous, i.e. $f \in C^1(I)$.
   (a) Show that if $f'(c) > 0$ then f is locally strictly increasing at c.
   (b) Is the converse of the above true? Prove your assertion.
   (c) Show that if f is locally increasing at c, that is, f is increasing on $B_r(c)$ for some $r > 0$ (with partial inequality), then $f'(c) \geq 0$.
   (d) Suppose $f'(c) \geq 0$. Is it true that f is locally increasing at c? Prove your assertion.
   (e) Suppose now f is differentiable but f' may not be continuous. Do statements in (a) - (d) still hold or not?

3. Let $f : I \to \mathbb{R}$ be a function. We say that f has the *Intermediate Value Property* if for all $x, y \in I$ such that $f(x) < f(y)$ and for all $t \in [f(x), f(y)]$, there exists $z \in [x, y]$ or $z \in [y, x]$ such that $f(z) = t$.

   (a) Show that if f is a continuous function, then f has the Intermediate Value Property.

   (b) Suppose f is differentiable on I. Suppose further that $x < y \in I$ such that $f'(x) < 0 < f'(y)$. Show that there exists $z \in (x, y)$ such that $f'(z) = 0$.

   (c) Suppose f is differentiable. Show that f' has the Intermediate Value Property.

   (d) Find a non-continuous function that has the Intermediate Value Property.

4. Let $f : [0, \infty)$ be a function. Let $t > 0$. We call a finite list of points $\{a_i\}_{i=-0}^{n}$ a partition of $[0, t]$ if $0 = a_0 < a_1 < \cdots < a_n := t$. We define

$$V_f(t) := V([0,t]) := \sup\{\sum_{i=1}^{n} |f(a_i) - f(a_{i-1})| : \{a_i\}_{i=0}^{n} \textit{ is a partition of } [0,t]\}$$

   Note that $V_f(t) \in [0, \infty]$ can take value $+\infty$. We call f to be of finite variation if $V_f(t) < \infty$ for all $t \geq 0$.

   (a) Suppose $f \in \mathcal{C}^1([0, \infty)$. Show that f is of finite variation.

   (b) Suppose $f \in \mathcal{C}^1([0, \infty)$. Show that f is the sum of two monotone functions.
   (*Hint: Show that $t \mapsto V_f(t)$ is an increasing function on* $[0, \infty)$).

5. Let $f : I \to \mathbb{R}$ be a function.

   (a) Show that f is convex if and only if for all finite list $\{\lambda_i\}_{i=1}^{n} \subset [0, 1]$ and $\{x_i\}_{i=1}^{n} \subset I$ such that $\sum_{i=1}^{n} \lambda_i = 1$, we have

$$f(\sum_{i=1}^{n} \lambda_i x_i) \leq \sum_{i=1}^{n} \lambda_i f(x_i)$$

   (b) Show that for all $n \in \mathbb{N}$ and $\{x_i\}_{i=1}^{n} \subset \mathbb{R}_{>0}$. We have $(x_1 \cdots x_n)^{1/n} \leq \dfrac{x_1 + \cdots + x_n}{n}$