Unless otherwise specified, $I \subset \mathbb{R}$ is an open interval.

Definition 1.1. Let $f : I \to \mathbb{R}$ be a function.

- We say that f is differentiable at $c \in I$ if $f'(c) := \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \in \mathbb{R}$ exists. In this case, we call $f'(c)$ the derivative of f at c
- We say that f is differentiable on I if f is differentiable at all $c \in I$. In that case we call $f' : I \to \mathbb{R}$ the derivative of f over I.

Practice Lv 1

1. Let $f : I \to \mathbb{R}$ be differentiable at $c \in I$. Show that f is continuous at c.

   Solution. Let $y \neq c \in I$. Then $f(y) = \frac{f(y)-f(c)}{y-c}(y - c) + f(c)$. It follows that

$$\lim_{y \to c} f(y) = \lim_{y \to c} \frac{f(y) - f(c)}{y - c} \lim_{y \to c}(y - c) + \lim_{y \to c} f(c) = f'(c) \cdot 0 + f(c) = f(c)$$

   Since c is clearly a cluster point of I, f is continuous at c.

2. Let $f : I \to \mathbb{R}$ be a function. Show that the following are equivalent:

   i. f is differentiable at $c \in I$

   ii. There exists $r > 0$ and a function $\phi : (c - r, c + r) \subset I \to \mathbb{R}$ such that ϕ is continuous at c and

$$f(x) - f(c) = \phi(x)(x - c)$$

   for all $x \in (c - r, c + r)$. We call such ϕ to be *locally defined* at c.

   Solution. (i)⇒(ii). Define $\phi : I \to \mathbb{R}$ by $\phi(x) := \frac{f(x)-f(c)}{x-c}$ for $x \neq c$ and $\phi(c) := f'(c)$. Then ϕ clearly satisfies the required condition. (ii)⇒(i). Suppose such ϕ exists. Then for $x \in B_r(c) \backslash \{c\}$ we have $\phi(x) = \frac{f(x)-f(c)}{x-c}$. Derivative of f at c exists due to the continuity of ϕ at c.

3. Let $f, g : I \to \mathbb{R}$ be differentiable at $c \in I$. Show that $f + g$ and fg are differentiable at c

   (a) by definition, and

   (b) by Q2

   Solution. See Lecture Note.

4. Let $f, g : I \to I$ be two functions such that f is differentiable at $c \in I$ and g is differentiable at $f(c) \in I$. Show that $g \circ f$ is differentiable at c.

   Solution. See Lecture Note.

Practice Lv 2

5. (P.171 Q10) Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) := \begin{cases} x^2 \sin(1/x^2) & x \neq 0 \\ 0 & x = 0 \end{cases}$. Show that

   (a) g is a differentiable function on $\mathbb{R}$.

   (b) g' is not bounded on $[-1, 1]$

   (You may assume the differentiability of sine functions)

   Solution. See HW1 solution.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := |\sin(x)|$. Find all points at which f is not differentiable. Explain your answer.

   Solution. f is not differentiable precisely at its zeros, that is, at $x = n\pi$ where $n \in \mathbb{Z}$. First we show that f is differentiable at x if $f(x) \neq 0$, that is, if $f(x) > 0$. Note by continuity of f (as f is composition of two continuous functions) we have $f > 0$ on $B_r(x)$ for some $r > 0$. Therefore $f(t) = \sin(t)$ on $B_r(x)$. The differentiability follows from that of the sine function.
   Now suppose $f(x) = 0$. Then $x = n\pi$ for some $n \in \mathbb{Z}$. Note that for $h > 0$, we have $\frac{f(x+h)-f(x)}{h} = \frac{|\sin(n\pi+h)|}{h} = \frac{|\sin(h)|}{h} = \frac{|\sin(h)|}{|h|}\frac{|h|}{h}$. As $\lim_{h\to 0} \frac{\sin(h)}{h} = 1$ but $\lim_{h\to 0} \frac{|h|}{h}$ does not exist. It follows that $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ does not exist. (Otherwise, contradiction arises as $\frac{|h|}{h} = \frac{f(x+h)-f(x)}{h}\frac{|h|}{|\sin(h)|}$)

7. Recall that $f : I \to \mathbb{R}$ is said to be Lipschitz function if there exists $L > 0$ such that $|f(x) - f(y)| \leq L|x - y|$ for all $x, y \in \mathbb{R}$. Let $f : I \to \mathbb{R}$ be a function.

   (a) Suppose f is Lipschitz and differentiable. Show that f' is bounded.

   (b) Can the Lipschitz assumption in part (a) be omitted? Explain your answer and give counter-examples if necessary.

   Solution. (a). Note that $\frac{|f(x)-f(y)|}{|x-y|} \leq L$ for all $x \neq y$ where L is a Lipschitz constant. Then it is clear that $|f'| \leq L$ (b). Yes. Consider $f(x) = x^2$ on $\mathbb{R}$. Then f' exists but is not bounded.

8. Let $f : I \to \mathbb{R}$ where I is bounded. Suppose f is differentiable and f' is uniformly continuous. Show that f is Lipschitz.
   (Hint: Show that f' is bounded first.)

   Solution. Note that I is bounded and uniform continuity preserves boundedness, so f' is bounded. Then it follows from Mean Value Theorem that f is Lipschitz as $|f(x) - f(y)| = |f'(c)||x - y| \leq \sup f'(I)|x - y|$ where $c \in (x, y)$ for all $x < y$.

9. Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function, that is, for all $x, y \in \mathbb{R}$ and $t \in [0, 1]$, we have

   $$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)$$

   (a) Let $x, y, z \in \mathbb{R}$ be such that $x < y < z$. Show that we have

   $$\frac{f(x) - f(y)}{x - y} \leq \frac{f(x) - f(z)}{x - z}$$

   (b) Show that for all $c \in \mathbb{R}$ the right limit $\lim_{x\to c^+} \frac{f(x)-f(c)}{x-c}$ exists; in particular it does **not** diverge to infinities.

   (c) Show that $\lim_{x\to c} f(x) = f(c)$ for all $c \in \mathbb{R}$.

   (Hint: It is better for you to first think about the meaning (e.g. graphically) of a convex function.)

   Solution. (a). We first write $y = tx + (1 - t)z$ for some $t \in [0, 1]$. It follows that $t = \frac{y-z}{x-z} \in [0, 1]$ as $x < y < z$. By convexity we then have $f(y) \leq tf(x) + (1 - t)f(z)$. Rearranging the terms gives the required inequality.
   (b). Fix $c \in \mathbb{R}$ and define $\phi_c(y) := \frac{f(c)-f(y)}{c-y}$ for all $y \in (c, \infty)$. Part (a) showed that ϕ_c in increasing. It is not hard to see that ϕ_c is bounded below by like $\frac{f(c)-f(t)}{c-t}$ where t can be any number $< c$. It then follows by considering infimum that $\lim_{y\to c^+} \phi_c(y)$ exists, which is the required limit.
   (c). Note that $f(y) = \phi_c(y)(y-c)+f(c)$ for all $y > c$. Hence $\lim_{y\to c^+} f(y) = f(c)$ by part (b). Therefore f is right continuous at c. Slightly modifying (a) and (b) gives that f is left continuous at c. Hence f is continuous at c. This shows that every convex function is continuous.