1 Unconditional Convergence

Definition 1.1. Let (x_n) be a sequence of real numbers. We say that $\sum x_n$ is unconditionally convergent if and only if for all permutations (bijections) $\sigma : \mathbb{N} \to \mathbb{N}$ we have $\sum_n x_{\sigma(n)}$ converges.

Remark. It can be shown that if $\sum x_n$ converges unconditionally, then $\sum x_{\sigma(n)}$ converges to the same limit for any permutation $\sigma \in S(\mathbb{N})$.

1.1 Quick Practice

1. Let (x_n) be a sequence of real numbers. We say that $\sum x_n$ converges absolutely if $\sum |x_n|$ converges.

   (a) Find an example of a series that converges but does not converge absolutely.

   (b) Show that $\sum x_n$ converges unconditionally if $\sum x_n$ converges absolutely.

   Solution. (a). $\sum_n \frac{(-1)^n}{n}$ and many more by the alternating series test. (b). See lecture notes.

2. Let (x_n) be a sequence of real numbers. Show that $\sum x_n$ converges unconditionally if and only if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all *finite* sets $F \subset [n, \infty) \cap \mathbb{N}$, we have $\left|\sum_{n\in F} x_n\right| < \epsilon$.

   Solution. ($\Leftarrow$). Let $\sigma \in S(\mathbb{N})$. We want to show that $\sum_n x_{\sigma(n)}$ converges. By Cauchy Criteria, it suffices to show that the partial sum $(\sum_{k=1}^n x_{\sigma(k)})$ is a Cauchy sequence. Now let $\epsilon > 0$. Then by assumption, there exists $N \in \mathbb{N}$ such that $\left|\sum_{n\in\mathbb{F}} x_n\right| < \epsilon$ for all finite sets $F \subset [n, \infty) \cap \mathbb{N}$. Now take K large enough such that $K > \pi^{-1}(1), \cdots, \pi^{-1}(N)$. Hence by the choice of K, for all $k \geq K$, we have $\sigma(k) \geq N$. Therefore, for all $n, m \geq K$, it follows that $\{\sigma(k)\}_{k=n}^m \subset [N, \infty) \cap \mathbb{N}$. Hence, we have $\left|\sum_{k=n}^m x_{\sigma(k)}\right| < \epsilon$.
   ($\Rightarrow$). Suppose not. Then there exists $\epsilon > 0$ such that for all $n \in \mathbb{N}$, there exists a finite subset $F \subset \mathbb{N}$ with $\min F \geq n$ such that $\left|\sum_{n\in F x_n}\right| \geq \epsilon$. From this, note that there exists a sequence of disjoint finite subsets (F_n) of $\mathbb{N}$ with the property that $\max F_n < \min F_{n+1}$ such that $\left|\sum_{k\in F_n} x_k\right| \geq \epsilon$ for all $n \in \mathbb{N}$. Now we define $\sigma \in S(\mathbb{N})$ by separately defining its action on $I_n := [\max F_{n-1} + 1, \max F_n]$ for all $n \geq 0$ with $F_0 := \{0\}$: on I_n, we define σ to place all elements of F_n to go after $I_n \backslash F_n$. With a careful observation on the definition of σ, it would imply that $\sum_n x_{\sigma(n)}$ does not have Cauchy partial sums as $\left|\sum_{k\in F_n} x_k\right| \geq \epsilon$ is the sum of consecutive elements in the defined permutation.

3. Let (x_n) be a sequence of real numbers.

   (a) Show that $\sum x_n$ converges unconditionally if and only if $\sum \epsilon_n x_n$ converges for all $(\epsilon_n) \in \{0, 1\}^{\mathbb{N}}$, that is (ϵ_n) is a sequence of signs. *Hint: Q2 could be useful*

   (b) Show that $\sum x_n$ converges unconditionally if and only if $\sum \epsilon_n x_n$ converges for all $(\epsilon_n) \in \{\pm 1\}^{\mathbb{N}}$, that is (ϵ_n) is a sequence of signs.

   (c) Hence, give an alternative proof that if $\sum x_n$ converges absolutely then $\sum x_n$ converges unconditionally.

   (d) Show that the converse of part (ii) is true: if $\sum x_n$ converges unconditionally, then $\sum x_n$ converges absolutely.

   Solution.

   (a) ($\Rightarrow$). Fix $(\epsilon_n) \in \{0, 1\}^{\mathbb{N}}$. We want to show that $\sum \epsilon_n x_n$ has Cauchy partial sums.
   Let $\epsilon > 0$. By Q2, there exists $N \in \mathbb{N}$ such that for all finite sets F with $\min F \geq N$, we have $\left|\sum_{n\in F} x_n\right| < \epsilon$. Note that when $n, m \geq N$ then we have $\sum_{k=n}^m \epsilon_k x_k = \sum_{\epsilon_k=1, k\in[n,m]} x_k$. It follows that $|\sum_{k=n}^m \epsilon_k x_k| \leq \epsilon$.
   ($\Leftarrow$). Suppose $\sum x_n$ does not converge absolutely. Then by similar arguments in Q2, there exists $\epsilon > 0$ and a sequence of disjoint finite subsets (F_n) of $\mathbb{N}$ with $\max F_n < \min F_{n+1}$s such that $\left|\sum_{k\in F_n} x_k\right| \geq \epsilon$. The result follows by considering (ϵ_n) such that $\sum_{k=\max F_{n-1}+1}^{\max F_n} \epsilon_k x_k = \sum_{k\in F_n} x_k$ for all $n \in \mathbb{N}$ where $F_0 := \{0\}$.

   (b) Algebraic computations with part (a).

   (c) Suppose $\sum x_n$ converges absolutely. It suffices to show that $\sum_n \epsilon_n x_n$ converges for all $\{_n\} \in \{\pm 1\}^{\mathbb{N}}$. It is clear because $|\epsilon_n x_n| = |x_n|$ and so $\sum \epsilon_n x_n$ converges absolultely.

   (d) Note that $\sum |x_n| = \sum \epsilon_n x_n$ for some $(\epsilon_n) \in \{\pm 1\}^{\mathbb{N}}$.

4. Let (x_n) be a sequence of real numbers. We say that (y_n) is a block sequence of (x_n) if there exists two sequences of positive real numbers $(p_n), (q_n)$ where $p_1 < q_1 < p_2 < q_2 < \cdots$ such that $y_n = \sum_{i=p_1}^{q_1} \alpha_i x_i$ where (α_i) is a sequence of real numbers.

   (a) show that $\sum x_n$ is not unconditionally converging if and only if there exists a block sequence (y_n) of (x_n) with coefficients $\{0,1\}$ (that is $(\alpha_i) \in \{0,1\}^{\mathbb{N}}$ in the definition) such that $\inf_n |y_n| > 0$.

   (b) Suppose $\sum x_n$ converges absolutely. Show that every block sequence with coefficients $\{0,1\}$ converges absolutely.

   Solution.

   (a) We would be using the fact that a series $\sum x_n$ converges unconditionally if and only if for all $(\epsilon_n) \in \{0,1\}^{\mathbb{N}}$, the series $\sum \epsilon_n x_n$ converges, that is, every subseries converges.
   $(\Rightarrow)$. Suppose (x_n) is not unconditionally converging. Then there exists a subseries (x_{n_k}) that does not converge. In particular, it is not Cauchy. Define $s_m := \sum_{k=1}^{m} x_{n_k}$. It follows from the non-Cauchiness that there exists $\epsilon > 0$ and two subsequences $(p_n), (q_n)$ satisfying $p_1 < q_1 < p_2 < q_2 < \cdots$ such that $\|s_{p_n} - s_{q_n}\| \geq \epsilon$. Define $F_j := \{n_k\}_{k=p_j}^{q_j}$ and $y_j := \sum_{i \in F_j} x_i$. Then it is clear that $\max F_j < \min F_{j+1}$ for all $j \in \mathbb{N}$ and so (y_j) is a block of (x_n). In addition it is clear from construction that $\inf_n \|y_n\| \geq \epsilon > 0$.
   $(\Leftarrow)$. Suppose such block sequence exists. Write $y_n := \sum_{j \in F_n} x_j$ where (F_n) are disjoint subsets of $\mathbb{N}$. Consider the subseries given by $(x_k)_{k \in \bigcup F_n}$. Then it is not hard to see that such subseries does not converge and so (x_n) does not converge unconditionally.

   (b) This is clear as $\sum_{k=1}^{n} |y_k| \leq \sum_{k=1}^{\infty} |x_k|$ for all $n \in \mathbb{N}$ and block sequence (y_n).

5. Let (x_n) be a sequence of real numbers. Show that $\sum x_n$ converges unconditionally if and only if for all bounded sequence of real numbers (λ_n) we have $\sum_n \lambda_n x_n$ converges.

   Solution. $(\Leftarrow)$. This follows as $\sum \epsilon_n x_n$ converges for all $(\epsilon_n) \in \{\pm\}^{\mathbb{N}}$.
   $(\Rightarrow)$. Note that $\sum_{k=1}^{n} |\lambda_n x_n| \leq \|(\lambda_n)\|_\infty \sum_{k=1}^{n} |x_n| \leq \|(\lambda_n)\|_\infty \sum_{k=1}^{\infty} |x_n|$. In addition $\sum x_n$ converges absolutely as it converges unconditionally (by Q3). It follows that $\sum_n \lambda_n x_n$ converges absolutely and so converges.

6. Name and verify a series that converges but is not unconditionally converging.

   Solution. See Q1.

7. Let X be a normed space.

   (a) Define suitable notions of unconditional converging series for X.

   (b) Suppose X is a Banach space, that is a normed space satisfying the Cauchy criteria: every Cauchy sequence converges. Show that the statements in $Q1, 2, 4$ under this more general setting are still true.

   (c) Following the previous part, show that $Q3a, b, c$ are still true under the more general setting. With the help of the internet, determine the condition that $Q3d$ is still valid under the more general setting and name the related Theorem.

   Solution. (a). Similar to the definition for $\mathbb{R}$ but the convergence is under norms instead of absolute values. (b). Similar proofs. Absolute convergence of a normed space is defined as the convergence of $\sum \|x_n\|$. (c). Q3abc are similar but the absolute-value-to-signs-proof is not immediate for Q3c. For Q3d, check the Dvoretzky-Rogers theorem on unconditional summability which states that unconditional convergence implies absolute convergence if and only if the Banach X is of finite dimension.