1 Unconditional Convergence

Definition 1.1. Let (x_n) be a sequence of real numbers. We say that (x_n) is unconditionally convergent if and only if for all permutations (bijections) $\sigma : \mathbb{N} \to \mathbb{N}$ we have $\sum_n x_{\sigma(n)}$ converges.

Remark. It can be shown that if $\sum x_n$ converges unconditionally, then $\sum x_{\sigma(n)}$ converges to the same limit for any permutation $\sigma \in S(\mathbb{N})$.

1.1 Quick Practice

1. Let (x_n) be a sequence of real numbers. We say that $\sum x_n$ converges absolutely if $\sum |x_n|$ converges.
   (a) Find an example of a series that converges but does not converge absolutely.
   (b) Show that $\sum x_n$ converges unconditionally if $\sum x_n$ converges absolutely.

2. Let (x_n) be a sequence of real numbers. Show that $\sum x_n$ converges unconditionally if and only if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all *finite* sets $F \subset [n, \infty) \cap \mathbb{N}$, we have $\left|\sum_{n \in F} x_n\right| < \epsilon$.

3. Let (x_n) be a sequence of real numbers.
   (a) Show that $\sum x_n$ converges unconditionally if and only if $\sum \epsilon_n x_n$ converges for all $(\epsilon_n) \in \{0, 1\}^{\mathbb{N}}$, that is (ϵ_n) is a sequence of signs. *Hint: Q2 could be useful*
   (b) Show that $\sum x_n$ converges unconditionally if and only if $\sum \epsilon_n x_n$ converges for all $(\epsilon_n) \in \{\pm 1\}^{\mathbb{N}}$, that is (ϵ_n) is a sequence of signs.
   (c) Hence, give an alternative proof that if $\sum x_n$ converges absolutely then $\sum x_n$ converges unconditionally.
   (d) Show that the converse of part (ii) is true: if $\sum x_n$ converges unconditionally, then $\sum x_n$ converges absolutely.

4. Let (x_n) be a sequence of real numbers. We say that (y_n) is a block sequence of (x_n) if there exists two sequences of positive real numbers $(p_n), (q_n)$ where $p_1 < q_1 < p_2 < q_2 < \cdots$ such that $y_n = \sum_{i=p_1}^{q_1} \alpha_i x_i$ where (α_i) is a sequence of real numbers.

   (a) show that $\sum x_n$ is not unconditionally converging if and only if there exists a block sequence (y_n) of (x_n) with coefficients $\{0, 1\}$ (that is $(\alpha_i) \in \{0, 1\}^{\mathbb{N}}$ in the definition) such that $\inf_n |y_n| > 0$.

   (b) Suppose $\sum x_n$ converges absolutely. Show that every block sequence with coefficients $\{0, 1\}$ converges absolutely.

5. Let (x_n) be a sequence of real numbers. Show that $\sum x_n$ converges unconditionally if and only if for all bounded sequence of real numbers (λ_n) we have $\sum_n \lambda_n x_n$ converges.

6. Name and verify a series that converges but is not unconditionally converging.

7. Let X be a normed space.

   (a) Define suitable notions of unconditional converging series for X.

   (b) Suppose X is a Banach space, that is a normed space satisfying the Cauchy criteria: every Cauchy sequence converges. Show that the statements in $Q1, 2, 4$ under this more general setting are still true.

   (c) Following the previous part, show that $Q3a, b, c$ are still true under the more general setting. With the help of the internet, determine the condition that $Q3d$ is still valid under the more general setting and name the related Theorem.