1 Convergence of Functions

Definition 1.1. Let $D \subset \mathbb{R}$. Let $f_n : D \to \mathbb{R}$ be a sequence of functions. Let $f : D \to \mathbb{R}$. We say that

- $f_n \to f$ pointwise if $\lim_n f_n(x) = f(x)$ for all $x \in D$
- $f_n \to f$ uniformly if $\lim_n \sup_{x\in D} |f_n(x) - f(x)| = 0$. In other words, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|f_n(x) - f(x)| < \epsilon$ for all $x \in D$.

Quick Practice

1. Let (f_n) be a sequence of functions on D.

   (a) Show that if $f_n \to f$ uniformly on D then $f_n \to f$ point-wise on D.

   (b) Show that $f_n \to f$ uniformly if and only if $f_n - f \to 0$ uniformly

   Solution. (a). Fix $x \in D$. Then $|f_n(x) - f(x)| \leq \sup_{x\in D} |f_n(x) - f(x)|$ for all $n \in \mathbb{N}$. The result follows as $n \to \infty$ by sandwich theorem. (b). This is just a rewriting of the definition.

2. Let (f_n) be a sequence of functions on $[0, 1]$.

   (a) Suppose $f_n \to f, g$ point-wise. Show that $f = g$ point-wise.

   (b) Suppose $f_n \to f, g$ uniformly. Show that $f = g$ point-wise.

   Solution. (a). Fix $x \in [0, 1]$. The $0 \leq |f(x) - g(x)| \leq |f(x) - f_n(x)| + |f_n(x) - g(x)|$ for all $n \in \mathbb{N}$ by the triangle inequality. The result follows from the squeeze theorem. (b) follows from (a) as uniform convergence implies point-wise convergence.

3. Let (f_n) be a sequence functions on D. Suppose (f_n) converges uniformly.

   (a) Suppose (f_n) is a sequence of bounded function. Show that (f_n) is *uniformly bounded*, that is, there exists $M > 0$ such that $\sup_{x\in D} |f_n(x)| < M$ for all $n \in \mathbb{N}$.

   (b) Suppose (f_n) is a sequence of uniformly continuous functions. Show that (f_n) is *equi-continuous*, that is, for all $\epsilon > 0$, there exists $\delta > 0$ such that whenever $x, y \in D$ are with $|x - y| < \delta$, that $|f_n(x) - f_n(y)| < \epsilon$ for all $n \in \mathbb{N}$.

   Solution. Write f the uniform limit. (a). Note that for all $x \in D$, we have $|f(x)| \leq |f_n(x) - f(x)| + |f_n(x)| \leq \sup_{x\in D} |f_n(x) - f(x)| + \|f_n\|_\infty$ for all $n \in \mathbb{N}$. Take $N \in \mathbb{N}$ such that $\sup_{x\in D} |f_n(x) - f(x)| \leq 1$ for $n \geq N$, then we have $\|f\|_\infty \leq 1 + \|f_N\|_\infty < \infty$ and so f is bounded. Next, observe that for all $n \geq N$, we have $|f_n(x)| \leq |f_n - f|_\infty + \|f\|_\infty \leq 1 + \|f\|_\infty$ and so $\|f_n\|_\infty \leq 1 + \|f\|_\infty$ for all $n \geq N$. Take $M := \max\{\|f_1\|_\infty, \cdots, \|f_N\|_\infty, 1 + \|f\|_\infty\}$ then M is a uniform bound for (f_n).
   (b). Let $\epsilon > 0$. Take $N \in \mathbb{N}$ such that $|f_n - f|_\infty < \epsilon$ for $n \geq N$. Let $\delta_i > 0$ be such that $|f_i(x) - f_i(y)| < \epsilon$ if $|x - y| < \delta_i$ for all $i \in \mathbb{N}$. Hence, when $|x - y| < \delta_N$, we have $|f(x) - f(y)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \leq 3\epsilon$. It follows that f is uniform continuous. Next, let $\delta := \min\{\delta_1, \cdots, \delta_N\}$. Then when $|x - y| < \delta$, we have $|f_i(x) - f_i(y)| < \epsilon$ when $i = 1, \cdots, N$. When $i > N$, we also have $|f_i(x) - f_i(y)| \leq |f(x) - f_i(x)| + |f(x) - f(y)| + |f(y) - f_i(y)| \leq \epsilon + 3\epsilon + \epsilon \leq 5\epsilon$. It follows that (f_n) is equi-continuous.

4. Suppose (f_n) is a sequence of functions over D that are continuous at $c \in D$. Further suppose $f_n \to f$ uniformly. Show that f is continuous at c.

   Solution. See lecture notes.

5. Let (f_n) be a sequence of functions on D. Suppose (f_n) does not converge to 0 uniformly.

   (a) Show that there exists $\epsilon > 0$, a subsequence (f_{n_k}) of (f_n) and a sequence of points (x_n) in D such that $|f_{n_k}(x_k)| \geq \epsilon$ for all $k \in \mathbb{N}$.

   (b) Show that $\overline{\lim}_n \sup_{x\in D} |f_n(x)| > 0$.

   Solution. (a). By the negation. There exists $\epsilon > 0$ such that $\sup_{x\in D} |f_{n_k}(x)| \geq \epsilon$ for some $n_k \geq k \in \mathbb{N}$ for all $k \in \mathbb{N}$. Fix $k \in \mathbb{N}$, by the definition of supremum, there exists $x_k \in D$ with $|f_{n_k}(x_k)| \geq \epsilon/2$. The result follows. (b). Note that $\lim_n \sup_{x\in D} |f_n(x)| = 0$ if and only if $\overline{\lim} \sup_{x\in D} |f_n(x)| = 0$ since $\sup_{x\in D} |f_n(x)| \geq 0$ for all $n \in \mathbb{N}$. It follows that $\overline{\lim} \sup_{x\in D} |f_n(x)| > 0$ as the limit is non-zero.

6. For each of the following domains $D \subset \mathbb{R}$ and sequences of functions $f_n : D \to \mathbb{R}$,

   (a). Find its point-wise limit f.

   (b). Determine whether $f_n \to f$ uniformly.

   i. $f_n(x) := \dfrac{x}{x+n}, D = [0,t], t > 0$

   ii. $f_n(x) := \dfrac{x}{x+n}, D = [0,\infty)$

   iii. $f_n(x) := \dfrac{x^2+nx}{n}, D = \mathbb{R}$

   iv. $f_n(x) := x^n, D = [0,t], t \in (0,1)$

   v. $f_n(x) := x^n, D = [0,1]$

   vi. $f_n(x) = x^{1+\frac{1}{2n+1}}, D = [-1,1]$

   vii. $f_n(x) := \dfrac{1}{n(1+x^2)}, D = \mathbb{R}$

   viii. $f_n(x) := \begin{cases} 1 & x = 1, 1/2, \cdots, 1/n \\ 0 & otherwise \end{cases}, D = [0,1]$

   ix. $f_n(x) := \begin{cases} x & x = 1, 1/2, \cdots, 1/n \\ 0 & otherwise \end{cases}, D = [0,1]$

   x. Enumerate $\mathbb{Q} \cap [0,1] = (q_n)$. Define $f_n(x) := \begin{cases} 1 & x = q_1, \cdots, q_n \\ 0 & otherwise \end{cases}, D = [0,1]$

   Solution. (i). $f(x) = 0$; the convergence is uniform: $|f_n(x)| \le f_n(t)$
   (ii). $f(x) = 0$; the convergence is not uniform: consider $x_n := n$.
   (iii). $f(x) = x$; the convergence is not uniform: consider $x_n := n$.
   (iv). $f(x) = 0$; the convergence is uniform: $|f_n(x)| \le t^n$ for all $n \in \mathbb{N}$ and $x \in [0,t]$.
   (v). $f = \mathbb{1}_{\{1\}}$; the convergence is not uniform: the point-wise limit is not continuous at 1 but the sequence is.
   (vi). $f(x) = |x|$; the convergence is uniform. The proof is as follows. We first show the uniform convergence on $[0,1]$. Let $\epsilon > 0$. Then

   $$|f_n(x) - f(x)| = \left|xx^{1/(2n+1)} - x\right| = \left|x(x^{1/(2n+1)-1})\right| = x(1 - x^{1/(2n+1)})$$

   When $x \in [0,\epsilon]$, we have $|f_n(x) - f(x)| \le \epsilon \cdot 1 = \epsilon$. When $x \in [\epsilon, 1]$, we have $|f_n(x) - f(x)| \le 1 \cdot (1 - \epsilon^{1/(2n+1)})$. Combining the two, it follows that

   $$\sup_{x\in[0,1]} |f_n(x) - f(x)| \le \epsilon + (1 - \epsilon^{1/(2n+1)})$$

   As $n \to \infty$, we have $\limsup_n \sup_{x\in[0,1]} |f_n(x) - f(x)| \le \epsilon + 1 - 1 = \epsilon$ (since $\epsilon^{1/(2n+1)} \to 1$). As ϵ is arbitrary, it follows that we have $\limsup_n \sup_{x\in[0,1]} |f_n(x) - f(x)| = 0$ as $\epsilon \to 0$ and so we can conclude that $\lim_n \sup_{x\in[0,1]} |f_n(x) - f(x)| = 0$. Similarly, the convergence is uniform on $[-1,0]$. Hence the convergence is uniform on the union $[-1,1] = [-1,0] \cup [0,1]$
   (vii). $f(x) = 0$; the convergence is uniform: $|f_n(x)| \le 1/n$ for all $x \in \mathbb{R}$.
   (viii). $f = \mathbb{1}_{\{\frac{1}{k}:k\in\mathbb{N}\}}$; the convergence is not uniform: note that $f - f_n = \mathbb{1}_{\{\frac{1}{k}:k\ge n+1\}}$ and so taking $x_n := (n+1)^{-1}$ will do.
   (ix). $f(x) = x\mathbb{1}_{\{\frac{1}{n}:n\in\mathbb{N}\}}(x)$. The convergence is uniform. Note that $|f_n(x) - f(x)| \le (n+1)^{-1}$.
   (x). $f(x) = \mathbb{1}_{\mathbb{Q}}$. The convergence is not uniform since f is not Riemann integrable but (f_n) is a sequence of integrable functions (over $[0,1]$).

7. Let (f_n) be a sequence of functions on D.

   (a) (Cauchy Criteria). Show that f_n converges uniformly if and only if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n, m \ge N$, we have $|f_n(X) - f_m(x)| < \epsilon$ for all $x \in D$

   (b) (Weierstrass M-test). Define $S_n(x) := \sum_{i=1}^{n} f_i(x)$ for all $x \in D$ and $n \in \mathbb{N}$. Suppose there exists a sequence of summable positive real numbers (M_n) (that is $\sum M_n < \infty$) such that $\sup_{x\in D} |f_n(x)| \le M_n$ for all $n \in \mathbb{N}$. Show that S_n converges uniformly.

   Solution. See Lecture notes.