Unless otherwise specified, if we write (a, b) or $[a, b]$, it is always the case that $a < b \in \mathbb{R}$.

1 Improper Integrals

Definition 1.1. Let $f : [a, b) \to \mathbb{R}$ be a function with $-\infty < a < b \leq \infty$. Suppose $f \in \mathcal{R}([a, c])$ for all $c \in [a, b)$. Then we define

$$\int_a^b f := \lim_{c \to b^-} \int_a^c f$$

to be the impropert integral of f over $[a, b)$ if the limit exists.

Quick Practice

1. Evaulate the improper integral $\int_0^\infty e^{-t} dt$

   Solution. Covered in Tutorial.

2. Suppose $f \in \mathcal{R}([0, t])$ for all $t > 0$. Show that $\int_0^\infty f$ exists if and only if for all $\epsilon > 0$, there exists $M > 0$ such that $s, t > M$ would imply $\left| \int_s^t f \right| < \epsilon$

   Solution. Covered in Tutorial and Lecture Note.

3. Suppose $f \in \mathcal{R}([0, t])$ for all $t > 0$. Show that if $\int_0^\infty |f|$ exists, then $\int_0^\infty f$ exists.

   Solution. Covered in Tutorial and lecture note.

4. Let $f : [0, \infty)$ be non-negative and $f \in \mathcal{R}([0, t])$ for all $t > 0$. Define $F(t) := \int_0^t f$ for all $t \geq 0$. Show that $\int_0^\infty f$ exists if and only if F is bounded on $[0, \infty)$.

   Solution. Covered in Tutorial.

5. Define $f(\alpha) := 1/\alpha$ and $g_t(\alpha) := e^{-\alpha t}$ for all $t, \alpha > 0$.

   (a) Find $g'(\alpha)$ and $g_t'(\alpha)$ for all $\alpha, t > 0$.

   (b) Show that $f(\alpha) = \int_0^\infty g_t(\alpha) dt$ for all $\alpha > 0$.

   (c) Is it true that $f'(\alpha) = \int_0^\infty g_t'(\alpha) dt$ for all $\alpha > 0$?

   Solution. (a) is easy. (b), (c): Use integration by part, a.k.a. FTC on the product rule.

6. (Integral Test). Let $f : [0, \infty)$ be a non-negative, decreasing function.

   (a) Show that for all $N > 1 \in \mathbb{N}$, we have

   $$\int_1^{N+1} f \leq \sum_{k=1}^{N} f(k) \leq \int_0^N f$$

   (b) Show that $\int_1^\infty f$ exists if and only if $\sum_{n=1}^\infty f(n)$ exists.

   (c) Show that

   $$\sum_{n=2}^{\infty} f(n) \leq \int_1^\infty f \leq \sum_{n=1}^{\infty} f(n)$$

   whenever the conditions in part (b) holds.

   (d) Show that $\sum_{n=1}^\infty \frac{1}{n^p} < \infty$ for all $p > 1$ and conclude that $\sum_{n=1}^\infty \frac{1}{n^2} \in [1, 2]$

   Solution. Covered in Tutorial.

7. Let $f(x) := \frac{x \log(x)}{1+x^2}$ for $x \in (0, 1]$ and $f(0) := 0$.

   (a) Show that there exists $M > 0$ and $t > 0$ such that $|x \log x| \leq M\sqrt{x}$ for all $x \in (0, t)$

   (b) Show that $\int_0^1 \frac{x \log x}{1+x^2}$ exists.

   Solution. (a). It suffices to show that $\lim_{x\to 0^+} \frac{x \log(x)}{\sqrt{x}} = \lim_{x\to 0^+} \sqrt{x} \log(x) = 0$. Note that we have the limit $\lim_{x\to 0^+} \frac{1}{\sqrt{x}} = \infty$ with $(1/\sqrt{x})' = \frac{-1}{2} x^{-3/2}$ for $x \in (0, 1)$, which is never 0. Hence, we can apply the L'Hospital Rule that

   $$\lim_{x\to 0^+} \frac{\log(x)}{1/\sqrt{x}} = \lim_{x\to 0^+} \frac{1/x}{-1/2x^{3/2}} = \lim_{x\to 0^+} -2\sqrt{x} = 0$$

   The requirement in question follows from the definition of limits.
   (b). It suffices to show that $f(x) \in \mathcal{R}([0, t])$ and $\mathcal{R}([t, 1])$ where t is the constant from part (a). The latter is true since f is continuous there. To show the former, it suffices to show that $|f| \in \mathcal{R}([0, t])$ by Q2. Note that $|f| \in \mathcal{R}[s, t]$ for all $s > 0$ as $|f|$ is continuous there. In addition we have

   $$\int_s^t |f| \leq \int_s^t M\frac{\sqrt{x}}{1+x^2} \leq \int_0^1 M\frac{\sqrt{x}}{(1+x^2)} < \infty$$

   The last term is finite because $\frac{\sqrt{x}}{1+x^2}$ is continuous on $[0, 1]$. It follows from Q4 that the improper integral $\int_0^t |f| < \infty$ and so $\int_0^t f$ exists.

8. (2016 - 17 Final Q2) Let $f : [1, \infty) \to \mathbb{R}$ be defined by $f(x) := \frac{\sin(x)}{x}$.

   (a) Show that $\int_1^\infty f(x)dx$ exists.

   (b) Show that $\int_1^\infty |f(x)|dx$ diverges.

   Solution. Covered in 2020-2021 2060B Tutorial 7, which is available here.

2 Convergence of Functions

Definition 2.1. Let $D \subset \mathbb{R}$. Let $f_n : D \to \mathbb{R}$ be a sequence of functions. Let $f : D \to \mathbb{R}$. We say that

- $f_n \to f$ pointwise if $\lim_n f_n(x) = f(x)$ for all $x \in D$
- $f_n \to f$ uniformly if $\lim_n \sup_{x \in D} |f_n(x) - f(x)| = 0$. In other words, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|f_n(x) - f(x)| < \epsilon$ for all $x \in D$.

Quick Practice

1. Let (f_n) be a sequence of functions on D. Show that if $f_n \to f$ uniformly on D then $f_n \to f$ point-wise on D.

   Solution. Covered in lecture.

2. Let (f_n) be a sequence of functions on D. Suppose (f_n) does not converge to 0 uniformly.

   (a) Show that there exists $\epsilon > 0$, a subsequence (f_{n_k}) of (f_n) and a sequence of points (x_n) in D such that $|f_{n_k}(x_k)| \geq \epsilon$ for all $k \in \mathbb{N}$.

   (b) Show that $\overline{\lim}_n \sup_{x \in D} |f_n(x)| > 0$.

   Solution. (a). Consider negation of uniform convergence. (b). Clearly follows from part (a).

3. Let (f_n) be a sequence of functions on $[0, 1]$.

   (a) Suppose $f_n \to f, g$ point-wise. Show that $f = g$ point-wise.

   (b) Suppose $f_n \to f, g$ uniformly. Show that $f = g$ point-wise.

   (c) Further assume $f_n \in \mathcal{R}([0, 1])$ and $f_n \geq 0$ pointwise. Suppose $f_n \to f, g$ in expectation, that is, $\lim \int_0^1 f_n = \int_0^1 f = \int_0^1 g$. Is it true that $f = g$ pointwise? What can you conclude about f and g ?

   Solution. (a), (b) are easy. (c). No. In fact f and g could be very different in a sense that we can find $f \geq 0$ to satisfy the convergence with g being negative on some intervals but very positive on others. Note that this convergence is weaker than $\int_0^1 |f_n - f| \to 0$.

4. Define $f_n(x) := \frac{x}{x+n}$ for all $x \in [0, \infty)$ and $n \in \mathbb{N}$.

   (a) Show that f_n converges point-wise on $[0, \infty)$ and find the limit.

   (b) Show that f_n does not converge uniformly on $[0, \infty)$

   (c) Show that f_n converges uniformly on $[0, t]$ for all $t > 0$.

   Solution. It is a possible HW question so we skip the solution.