1 (P. 215 Q12). Let $g(x) := \begin{cases} \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$ for all $x \in [0,1]$. Show that $g \in \mathcal{R}([0,1])$

Solution. First, note $g \in \mathcal{R}([t,1])$ for all $t > 0$ as g is continuous on $[t,1]$ for all $t > 0$. Now we prove the assertion by definition. Let $1 > \epsilon > 0$. Since $g \in \mathcal{R}([\epsilon,1])$, there exists a partition P such that $U(f,P) - L(f,P) < \epsilon$. Now consider $Q := \{0\} \cup P \subset [0,1] =: \{x_i\}_{i=1}^n$, which is a partition of $[0,1]$. Then we have

$$U(g,Q) - L(g,Q) = \sum_{i=1}^n \operatorname{diam} g([x_i, x_{i-1}])(x_i - x_{i-1}) = \operatorname{diam}(g([0,\epsilon]))\epsilon + U(g,P) - L(g,P) \leq 2\epsilon + \epsilon < 3\epsilon$$

as $\operatorname{diam}(g([0,\epsilon])) = \sup_{x,y\in[0.\epsilon]} |g(x) - g(y)| \leq 2$. It follows from definitions that $g \in \mathcal{R}([0,1])$.

Common Mistake. A number of you has argued as follows: fix $\epsilon > 0$. Since g is continuous on $[\epsilon, 1]$, it remains to show that $g \in \mathcal{R}([0,\epsilon])$. The latter is true because for any partition $P \subset [0,\epsilon]$, we have $U(g,P) - L(g,P) \leq 2\epsilon$.

This argument is NOT correct. To show $g \in \mathcal{R}([0,\epsilon])$ you should instead show that for all $\delta > 0$, there exists $P \subset [0,\epsilon]$ such that $U(g,P) - L(g,P) < \delta$. In the previous argument, ϵ has been fixed so you should consider some other arbitrarily defined variables.

2 (P. 215 Q18). Let $f : [a,b] \to \mathbb{R}$ be continuous. Define $M_n := (\int_a^b f^n)^{1/n}$. Suppose $f \geq 0$ on $[a,b]$. Show that $\lim M_n = \sup\{f(x) : x \in [a,b]\}$.

Solution. Write $\|f\|_\infty := \sup\{f(x) : x \in [a,b]\}$. Note that the assertion is clear if $\|f\|_\infty = 0$ (which implies $f \equiv 0$ on $[a,b]$ constantly.
Next we consider the case where $\|f\|_\infty = 1$. The strategy is to find some lower and upper bounds for (M_n) so that the sandwich theorem can be used. We would be showing the following claim:
Claim. For all $\epsilon > 0$ there exists $c < d \in \mathbb{R}$ such that $[c,d] \subset [a,b]$ and

$$(1-\epsilon)(d-c)^{1/n} \leq M_n \leq (b-a)^{1/n}$$

Proof of claim. Note that we have $f^n \leq 1$ pointwise for all $n \in \mathbb{N}$. Therefore, $\int_a^b f_n \leq \int_a^b 1 = b - a$ for all $n \in \mathbb{N}$. Hence, $M_n \leq (b-a)^{1/n}$ for all $n \in \mathbb{N}$. For the lower bound, we fix $\epsilon > 0$. Note that by the extreme value theorem, $f(t) = 1 = \|f\|_\infty$ for some $t \in [0,1]$. By continuity at t, we conclude that there exists an non-empty interval $I \subset [0,1]$ such that $f > 1 - \epsilon$ on I. In particular, we can choose some smaller compact intervals $[c,d] \subset I$. Then $f > 1 - \epsilon$ on $[c,d]$. It follows from the non-negativity of f that we have $\int_a^b f^n \geq \int_c^d f^n \geq \int_c^d (1-\epsilon)^n = (1-\epsilon)(d-c)$. It follows that we have $(1-\epsilon)(d-c)^{1/n} \leq M_n$ for all $n \in \mathbb{N}$. Hence the claim is proved.

To finish the case for $\|f\|_\infty = 1$, we fix $\epsilon > 0$. Then by considering $n \to \infty$ for the inequality in the claim, we have

$$1 - \epsilon = (1-\epsilon)\liminf (d-c)^{1/n} \leq \liminf M_n \leq \limsup M_n \leq \limsup (b-a)^{1/n} = 1$$

As $\epsilon > 0$ is arbitrary, it follows that we have $1 \leq \liminf M_n \leq \limsup M_n \leq 1$ as $\epsilon \to 0$. Hence, we have $\lim M_n = 1 = \|f\|_\infty$.

Finally, for the general case where $\|f\|_\infty \neq 0$. We can define $g := f/\|f\|_\infty$. Then it follows that $\|g\|_\infty = 1$. Hence, by the previous case, we have $\lim_n (\int_a^b g^n)^{1/n} = 1$. Note that

$$(\int_a^b g^n)^{1/n} = (\int_a^b \frac{f^n}{\|f\|_\infty^n})^{1/n} = \frac{1}{\|f\|_\infty} M_n$$

for all $n \in \mathbb{N}$. It follows that $\lim_n M_n = \|f\|_\infty$.

Common Mistake. Surprisingly, many of you did the question with the correct idea. Keep it up! Nonetheless, only a few of you have correctly used lim inf and lim sup to conclude $\lim M_n = \|f\|_\infty$. Note that we have to take first $n \to \infty$ and $\epsilon \to 0$ for the question. In addition, the sandwich theorem cannot be applied since the limit of the upper and lower bounds are not equal when $n \to \infty$.

3 (P. 225 Q21)**.** Let $f, g \in \mathcal{R}([a, b])$.

(a). Let $t \in \mathbb{R}$. Show that $\int_a^b (tf \pm g)^2 \geq 0$.

(b). Using (a), show that

$$2\left|\int_a^b fg\right| \leq t\int_a^b f^2 + \frac{1}{t}\int_a^b g^2$$

for all $t > 0$

(c). Suppose $\int_a^b f^2 = 0$. Show that $\int_a^b fg = 0$

(d). Prove the Cauchy-Bunyakovsky-Schwarz Inequality (or simply Schwarz Inequality):

$$\left|\int_a^b fg\right|^2 \leq (\int_a^b |fg|)^2 \leq (\int_a^b f^2)(\int_a^b g^2)$$

Solution.

(a). Note that $(tf \pm g)^2 \geq 0$ point-wise. It follows from monotonicity of integrals that $\int_a^b (tf \pm g)^2 \geq \int_a^b 0 = 0$ for all $t \in \mathbb{R}$

(b). Fix $t > 0$. Note that from part (a), we have

$$0 \leq \int_a^b (tf \pm g)^2 = t^2\int_a^b f^2 \pm 2t\int_a^b fg + \int_a^b g^2$$

It follows that we have

$$\pm 2\int_a^b fg \leq t\int_a^b f^2 + \frac{1}{t}\int_a^b g^2$$

since $t > 0$. Hence we have $\left|2\int_a^b fg\right| \leq t\int_a^b f^2 + \frac{1}{t}\int_a^b g^2$.

(c). Suppose $\int_a^b f^2 = 0$. It follows that we have

$$2\left|\int_a^b fg\right| \leq \frac{1}{t}\int_a^b g^2$$

for all $t > 0$. By $t \to \infty$, it follows that we have $\left|\int_a^b fg\right| = 0$ and so $\int_a^b fg = 0$.

(d). Note that we always have $\left|\int_a^b fg\right| \leq \int_a^b |fg|$ by triangle inequality. Hence the first inequality follows by taking squares. For the second inequality, first we suppose $\int_a^b f^2 \neq 0$. Note that by part (a), for all $t \in \mathbb{R}$, we have

$$F(t) := t^2\int_a^b f^2 + 2t\int_a^b |fg| + \int_a^b g^2 \geq 0$$

Note that $F : \mathbb{R} \to \mathbb{R}$ is a quadratic polynomial with positive leading coefficient. Since $F \geq 0$ everywhere, it never has distinct roots. Therefore by elementary algebra, we have $\Delta \leq 0$ where Δ is the discriminant of F. It follows that

$$4(\int_a^b |fg|)^2 - 4(\int_a^b f^2)(\int_a^b g^2) \leq 0$$

which implies the second inequality. Now suppose instead $\int_a^b f^2 = 0$. Then $\int_a^b |f|^2 = 0$. By part (c), it follows that $\int_a^b |f||g| = \int_a^b |fg| = 0$. Hence, the second inequality is also satisfied in this case.

Common Mistake. A number of you cannot do part (d), which is quite tricky in my opinion. Moreover many of you missed the case where $\int_a^b f^2 = 0$. In addition to the discriminant method above, other methods include:

- Take $t = (\int_a^b g^2 / \int_a^b f^2)^{1/2}$ or $t = (\int_a^b |fg|)/\int_a^b f^2$
- Take $t = 1$ and proceed as the proof of Tutorial 3 P.2 Q3, that is, by taking $t = 1$, we have $\int_a^b |fg| \leq \frac{1}{2}\int_a^b f^2 + \frac{1}{2}\int_a^b g^2$. Then we first consider the case where $\int_a^b f^2 = \int_a^b g^2 = 1$. Next, we consider in the general case that $f' := f/(\int_a^b f^2)^{1/2}$ and $g' := g/(\int_a^b g^2)^{1/2}$. This is a so-called normalization argument.