In the following, unless otherwise specificed, if $f : A \to \mathbb{R}$ is a bounded function on $A \subset \mathbb{R}$, then we denote $\|f\|_\infty := \sup\{|f(x)| : x \in A\}$.

1 (P. 252 Q12). Show that $\lim_n \int_1^2 e^{-nx^2} dx = 0$

Solution. Write $f_n(x) := e^{-nx^2}$ for all $x \in [1, 2]$. Note that f_n is non-negative and decreasing on $[1, 2]$ (e.g. by considering its derivatives). Hence, we have $0 \leq f_n(x) \leq f_n(1) = e^{-n}$ for all $x \in [1, 2]$ and $n \in \mathbb{N}$. It follows clearly that $\|f_n\|_\infty \leq e^{-n}$ and so $\lim_n \|f_n\|_\infty = 0$, that is, $f_n \to 0$ uniformly on $[1, 2]$. Hence, we have $\lim_n \int_1^2 f_n(x)dx = \int_1^2 \lim_n f_n(x)dx = \int_1^2 0dx = 0$ by the exchange of integrals and limits.

Common Mistake. A number of you claimed the interchange of the limit and integral here by BCT (Bounded Convergence Theorem); this is not suitable as BCT concerns about Lebesgue integrable functions instead of Riemann integrable functions. In particular, this is not suitable in this course.

2 (P. 252 Q14). Let $f_n(x) := \frac{nx}{1+nx}$ for $x \in [0, 1]$ and $n \in \mathbb{N}$.

i. Show that f_n converges non-uniformly to some integrable function f.

ii. Show that $\int_0^1 f(x)dx = \lim_n \int_0^1 f_n(x)dx$

Solution.

i. We claim that f_n converges point-wise to $f = \mathbb{1}_{(0,1]} := \begin{cases} 1 & x > 0 \\ 0 & x = 1 \end{cases}$ on $[0, 1]$. The case for $x = 0$ is clear. Suppose $x \in (0, 1]$. Then $f_n(x) = \frac{x}{\frac{1}{n}+x} \to \frac{x}{x} = 1$ as $n \in \infty$. Hence the result follows.
Next, we show that the convergence is not uniform on $[0, 1]$. Take $x_n := \frac{1}{n}$ for all $n \in \mathbb{N}$. Then $f_n(x_n) - f(x_n) = \frac{1}{2} - 1 = \frac{-1}{2}$. Hence, $\|f_n - f\|_\infty \geq \frac{1}{2} > 0$ for all $n \in \mathbb{N}$. It follows that f_n does not converge to f uniformly.

ii. Note that for all $n \in \mathbb{N}$ we have by FTC that

$$\int_0^1 f_n(x)dx = \int_0^1 \frac{nx}{1+nx}dx = \int_0^1 1 - \frac{1}{1+nx}dx = [x - \frac{1}{n}\log(1+nx)]_0^1 = 1 - \frac{1}{n}\log(1+n)$$

Note that $\lim_n \frac{1}{n}\log(1+n) = \lim_n \frac{1}{1+n} = 0$ by the L'Hospital rule on $f(x) := \log(1+x)$ and $g(x) := x$ under the form $\lim_{x\to\infty} \frac{f(x)}{g(x)}$, followed by the sequential crtieria. Hence, $\lim_n \int_0^1 f_n(x)dx = 1$. Meanwhile, we have $\int_0^1 f(x)dx = \int_0^1 \mathbb{1}_{(0,1]} = 1$ clearly. The result follows.

Remark. The non-uniform convergence in part (i) could be done by a 1-line proof by noting that f is not continuous.

3 (P. 252 Q19). Let $f_n(x) := \frac{x}{n}$ for $x \geq 0$ and $n \in \mathbb{N}$.

i. Show that (f_n) is a decreasing sequence of continuous functions that converges to a continuous f.

ii. Show that the convergence is non uniform.

Solution.

i. Note that (f_n) is a sequence of polynomial functions and so is a sequence of continuous functions. Now fix $n \leq m \in \mathbb{N}$. Then $f_n(x) = \frac{x}{n} \geq \frac{x}{m} = f_m(x)$ for all $x \geq 0$ clearly. It follows that $f_n \geq f_m$ point-wise. Therefore, (f_n) is a sequence of decreasing function. Finally, $\lim f_n(x) = \lim_n \frac{x}{n} = 0$ for all $x \geq 0$ clearly. Therefore $f(x) = 0$ for all $x \geq 0$, which is a continuous function.

ii. Take $x_n := n$ for all $n \in \mathbb{N}$. Then $|f_n(x_n) - f(x_n)| = f_n(x_n) = \frac{n}{n} = 1 > 0$ for all $n \in \mathbb{N}$. It follows that the convergence is not uniform.