# MATH2060 Solution 5

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## 7.3 Q10

Set $F(x) = \int_a^x$ on $[a, b]$. Since $f$ is continuous, we know $F$ is differentiable and $F'(x) = f(x)$ for any $x \in [a, b]$. Note that $G(x) = F(\nu(x))$. By chain rule,

$$G'(x) = F'(\nu(x))\nu'(x) = f(\nu(x))\nu'(x).$$

## 7.3 Q11

We can use the results from Q10.
(a) Note that $f(x) = \frac{1}{1+x^3}$ is continuous and $\nu(x) = x^2$ is differentiable. Hence,

$$F'(x) = f(\nu(x))\nu'(x) = \frac{2x}{1+x^6}.$$

(b) Note that $f(x) = \sqrt{1+t^2}$ is continuous and $\nu(x) = x^2$ is differentiable. Moreover, we can write $F(x) = \int_0^x \sqrt{1+t^2}dt - \int_0^{x^2} \sqrt{1+t^2}dt$. Hence we have

$$F'(x) = \sqrt{1+x^2} - f(\nu(x))\nu'(x) = \sqrt{1+x^2} - 2x\sqrt{1+x^4}.$$

## 7.3 Q13

When $x \in [0, 2)$, we have $G(x) = \int_0^x -1dt = -x$. When $x \in [2, 3]$, $G(x) = \int_0^2 -1dt + \int_2^x 1dt = x - 4$. Note that $G$ is not differentiable at $x = 2$ as

$$\lim_{h\to 0^+} \frac{G(2+h) - G(2)}{h} = 1 \neq -1 = \lim_{h\to 0^-} \frac{G(2+h) - G(2)}{h}.$$

## 7.3 Q16

Set $G(x) = \int_0^x f$. Note that $\int_0^x f = \int_x^1 f = \int_0^1 f - \int_0^x f$. Hence $G(x) = \frac{1}{2}\int_0^1 f$, which is a constant function. Since $f$ is continuous, $G$ is differentiable and $f(x) = G'(x) \equiv 0$ .

## 7.4 Q1

(a)

$$L(f;\mathcal{P}) = \sum m_k(x_k - x_{k-1}) = 0(0-(-1)) + 0(1-0) + 1(2-1) = 1. \quad (1)$$

$$U(f;\mathcal{P}) = \sum M_k(x_k - x_{k-1}) = 1(0-(-1)) + 1(1-0) + 2(2-1) = 4. \quad (2)$$

(b)

$$L(f;\mathcal{P}) = \sum m_k(x_k - x_{k-1}) = \frac{1}{2}(\frac{1}{2} + 0 + 0 + \frac{1}{2} + 1 + \frac{3}{2}) = \frac{7}{4}. \quad (3)$$

$$U(f;\mathcal{P}) = \sum M_k(x_k - x_{k-1}) = \frac{1}{2}(1 + \frac{1}{2} + \frac{1}{2} + 1 + \frac{3}{2} + 2) = \frac{13}{4}. \quad (4)$$

## 7.4 Q5

Since $f, h$ are Darboux integrable and $\int_a^b f = \int_a^b h$, we have

$$L(f) = U(f) = \int_a^b f = \int_a^b h = L(h) = U(h).$$

Since $f(x) \leq g(x) \leq h(x)$ for any $x \in [a, b]$, we know $L(f) \leq L(g)$ and $U(g) \leq U(h)$. This implies $L(g) = U(g) = \int_a^b f$ and hence $g$ is Darboux integrable with $\int_a^b g = \int_a^b f$.