$\underline{\text{Thm 9.3.4}}$ ($\underline{\text{Dirichlet's Test}}$)

If $\begin{cases} \bullet\ (x_n) \ \underline{\text{decreasing}} \ \& \ \underline{\lim x_n = 0} \\ \bullet\ (s_n = \sum_{k=1}^{n} y_k) \text{ are } \underline{\text{bounded}}, \end{cases}$

then $\sum x_n y_n$ is $\underline{\text{convergent}}$.

$\underline{\text{Pf}}$: (s_n) bdd $\Rightarrow$ $\exists B>0$ s.t. $|s_n| \le B$, $\forall n \in \mathbb{N}$.

Then Abel's Lemma (Thm 9.3.3) $\Rightarrow$ for $m>n$,

$$\left|\sum_{k=n+1}^{m} x_k y_k\right| \le |x_m s_m - x_{n+1} s_n| + \sum_{k=n+1}^{m-1} |x_k - x_{k+1}|\,|s_k|$$

$$\le (x_m + x_{n+1})B + \sum_{k=n+1}^{m-1} (x_k - x_{k+1})B$$

$\left(\begin{array}{c} \text{since } x_n \text{ decreasing} \\ x_k - x_{k+1} \ge 0 \end{array} \right)$

$$= B\left[(x_m + x_{n+1}) + (x_{n+1} - x_m)\right]$$

$$= 2x_{n+1}B \longrightarrow 0 \quad \text{as } n \to \infty$$

$\left(\therefore \ \forall \varepsilon > 0, \exists K \in \mathbb{N} \text{ s.t. if } m > n \ge K, \ \left|\sum_{k=n+1}^{m} x_n y_n\right| < \varepsilon \right)$

By Cauchy Criterion (Thm 3.7.4), $\sum x_n y_n$ is convergent. #

Thm 9.3.5 (**Abel's Test**)

If $\begin{cases} \bullet\ (x_n) \text{ convergent monotone sequence} \\ \bullet\ \sum y_n \text{ convergent} \end{cases}$

Then $\sum x_n y_n$ is also convergent.

(multiplying convergent monotone coefficients to a convergent series results in a convergent series.)

Pf: Case 1: (x_n) decreasing & $\lim x_n = x$

Let $u_n = x_n - x$, $\forall n \in \mathbb{N}$.

Then (u_n) decreasing & $u_n \to 0$.

Now $\sum y_n$ converges $\Rightarrow$ partial sum of $\sum y_n$ are bounded

$\therefore$ Dirichlet's Test (Thm 9.3.4) $\Rightarrow$ $\sum u_n y_n$ is convergent.

Hence $\sum x_n y_n = \sum (u_n + x) y_n = \sum u_n y_n + x \sum y_n$

is also convergent.

Case 2 (x_n) increasing, $x = \lim x_n$.

Let $v_n = x - x_n$, $\forall n \in \mathbb{N}$.

Then (v_n) decreasing & $v_n \to 0$ as $n \to \infty$.

Now $\sum y_n$ converges $\Rightarrow$ partial sum of $\sum y_n$ are bounded

$\therefore$ Dirichlet's Test $\Rightarrow$ $\sum v_n y_n$ convergence.

$\Rightarrow$ $\sum x_n y_n = \sum (x - v_n) y_n = x \sum y_n - \sum v_n y_n$

is also convergent. ※

<u>Eg 9.3.6</u> (a) Recall $2\sin\frac{1}{2}x\,(\cos x+\cdots+\cos nx)=\sin(n+\frac{1}{2})x-\sin\frac{1}{2}x$ (Ex)

If x is fixed and $x\neq 2k\pi$, $\forall k=\cdots -2,-1,0,1,2,\cdots$

Then $\underbrace{|\cos x+\cdots+\cos nx|}_{\text{partial sum of } \sum \cos nx} = \dfrac{|\sin(n+\frac{1}{2})x-\sin\frac{1}{2}x|}{2|\sin\frac{1}{2}x|} \le \underbrace{\dfrac{1}{|\sin\frac{1}{2}x|}}_{\text{bound indep. of } n}$, $\forall n\in\mathbb{N}$

$\therefore$ For a fixed $x\neq 2k\pi$, Dirichlet's Test $\Rightarrow$

$\sum_{n=1}^{\infty} a_n\cos nx$ converges, provided (a_n) is decreasing & $\lim a_n=0$.

(b) Similarly, from

$$2(\sin\tfrac{1}{2}x)(\sin x+\cdots+\sin nx)=\cos\tfrac{1}{2}x-\cos(n+\tfrac{1}{2})x, \quad \forall n\in\mathbb{N}$$

we have, for $x\neq 2k\pi$,

$$\underbrace{|\sin x+\cdots+\sin nx|}_{\text{partial sum of } \sum \sin nx} \le \underbrace{\frac{1}{|\sin\frac{1}{2}x|}}_{\text{bound indep. of } n} \quad \forall n\in\mathbb{N}$$

$\Rightarrow$ $\sum_{n=1}^{\infty} a_n\sin nx$ converges for $x\neq 2k\pi$ provided (a_n) decreasing and $\lim a_n=0$.

§9.4 Series of Functions

Def 9.4.1

If (f_n) is a seq. of functions defined on $D \subseteq \mathbb{R}$ (with $\mathbb{R}$-value), then the sequence of partial sums (S_n) of the infinity series of functions $\sum f_n$ is defined by

$$S_n(x) = \sum_{k=1}^{n} f_k(x), \quad \forall x \in D$$

- If (S_n) converges to a function f on D, then we say that the infinite series of functions $\sum f_n$ converges to f on D.

(usually write $f(x) = \sum_{n=1}^{\infty} f_n(x)$, $f = \sum_{n=1}^{\infty} f_n$, or $f = \sum f_n$)

- If $\sum |f_n(x)|$ converges $\forall x \in D$, then we say that $\sum f_n$ is absolutely convergent on D.
- If $S_n \rightrightarrows f$ (uniformly) on D, then we say that $\sum f_n$ is uniformly convergent on D, or $\sum f_n$ converges to f uniformly on D

Using $S_n \rightrightarrows f \Leftrightarrow \sum f_n$ converges to f uniformly, Thm 8.2.2, Thm 8.2.3 & Thm 8.2.4 imply the following theorems immediately:

Thm 9.4.2 If
- f_n continuous on D, $\forall n \in \mathbb{N}$
- $\sum f_n$ converges to f uniformly on D

Then f is continuous on D

(Pf : Applying Thm 8.2.2 to $S_n \rightrightarrows f$.)

Thm 9.4.3 If
- $f_n \in R[a,b]$, $\forall n \in \mathbb{N}$ $(a<b \in \mathbb{R})$
- $\sum f_n$ converges to f uniformly on $[a,b]$

Then $f \in R[a,b]$ and

$$\int_a^b f = \sum_{n=1}^{\infty} \int_a^b f_n$$

$$\left(\int_a^b \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int_a^b f_n \right)$$

(Pf : Applying Thm 8.2.4 to $S_n \rightrightarrows f$.)

Thm 9.4.4 If
- $f_n : [a,b] \to \mathbb{R}$, $n \in \mathbb{N}$ $(a<b \in \mathbb{R})$,
- f_n' exists on $[a,b]$, $\forall n \in \mathbb{N}$,
- $\exists x_0 \in [a,b]$ s.t. $\sum f_n(x_0)$ converges,
- $\sum f_n'$ converges uniformly on $[a,b]$.

Then $\exists f : [a,b] \to \mathbb{R}$ such that
- $\sum f_n$ converges to f uniformly on $[a,b]$,
- f' exists and $f' = \sum_{n=1}^{\infty} f_n'$

(Pf : Applying Thm 8.2.3 to S_n with S_n' converges uniformly etc.)

Tests for Uniform Convergence

Thm 9.4.5 (Cauchy Criterion)

$\sum f_n$ is uniformly convergent on D $\iff$

$\forall \varepsilon > 0$, $\exists K(\varepsilon) \in \mathbb{N}$ such that

if $m > n \geq K(\varepsilon)$, then $|f_{n+1}(x) + \cdots + f_m(x)| < \varepsilon$, $\forall x \in D$.

(Pf: Applying Cauchy Criterion for Uniform Convergence (Thm 8.1.10) to S_n and observing that

$$S_m(x) - S_n(x) = f_{n+1}(x) + \cdots + f_m(x).$$)

Thm 9.4.6 (Weierstrass M-Test)

If
- $|f_n(x)| \leq M_n$, $\forall x \in D$, $\forall n \in \mathbb{N}$
- $\sum M_n$ is convergent

then $\sum f_n$ is uniformly convergent on D

Pf: $\sum M_n$ convergent & $M_n \geq 0 \Rightarrow$

$\forall \varepsilon > 0$, $\exists K(\varepsilon) \in \mathbb{N}$ such that

if $m > n \geq K(\varepsilon)$, then $M_{n+1} + \cdots + M_m < \varepsilon$. (Thm 3.7.4)

Hence $|f_{n+1}(x) + \cdots + f_m(x)| \leq M_{n+1} + \cdots + M_m < \varepsilon$.

Cauchy Criterion (Thm 9.4.5) $\Rightarrow$ $\sum f_n$ converges uniformly on D #