Properties of Integral

Thm 7.1.5 Suppose $f, g \in \mathcal{R}[a,b]$. Then

(a) $kf \in \mathcal{R}[a,b]$, $\forall k \in \mathbb{R}$ and
$$\int_a^b kf = k \int_a^b f$$

(b) $f+g \in \mathcal{R}[a,b]$ and
$$\int_a^b (f+g) = \int_a^b f + \int_a^b g$$

(c) $f(x) \le g(x)$ $\forall x \in [a,b] \Rightarrow \int_a^b f \le \int_a^b g$.

Pf: (a) Ex. (Similar to the proof of (b) & easier)

(b) $f, g \in \mathcal{R}[a,b] \Rightarrow$

$\forall \varepsilon > 0$, $\exists \delta_1 > 0$ s.t. $\left| S(f, \dot{\mathcal{P}}) - \int_a^b f \right| < \varepsilon$, $\forall \dot{\mathcal{P}}$ with $\|\dot{\mathcal{P}}\| < \delta_1$

& $\exists \delta_2 > 0$ s.t. $\left| S(g, \dot{\mathcal{P}}) - \int_a^b g \right| < \varepsilon$, $\forall \dot{\mathcal{P}}$ with $\|\dot{\mathcal{P}}\| < \delta_2$.

Also note that for any $\dot{\mathcal{P}} = \{[x_{i-1}, x_i], t_i\}_{i=1}^n$,
$$\begin{aligned}
S(f+g; \dot{\mathcal{P}}) &= \sum_{i=1}^n (f+g)(t_i)(x_i - x_{i-1}) \\
&= \sum_{i=1}^n f(t_i)(x_i - x_{i-1}) + \sum_{i=1}^n g(t_i)(x_i - x_{i-1}) \\
&= S(f; \dot{\mathcal{P}}) + S(g; \dot{\mathcal{P}})
\end{aligned}$$

Then $\forall \dot{\mathcal{P}}$ with $\|\dot{\mathcal{P}}\| < \delta = \min\{\delta_1, \delta_2\}^{(70)}$, we have

$$\left| S(f+g;\dot{\mathcal{P}}) - \left(\int_a^b f + \int_a^b g\right) \right|$$

$$\leq \left| S(f;\dot{\mathcal{P}}) - \int_a^b f \right| + \left| S(g;\dot{\mathcal{P}}) - \int_a^b g \right|$$

$$< \varepsilon + \varepsilon = 2\varepsilon$$

Since $\varepsilon > 0$ is arbitrary, we've proved that

$$f+g \in \mathcal{R}[a,b] \text{ and } \int_a^b (f+g) = \int_a^b f + \int_a^b g.$$

(c) As in (b), we conclude, $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. for $\dot{\mathcal{P}}$ with $\|\dot{\mathcal{P}}\| < \delta$,

$$\left| S(f;\dot{\mathcal{P}}) - \int_a^b f \right| < \varepsilon \quad \text{and} \quad \left| S(g;\dot{\mathcal{P}}) - \int_a^b g \right| < \varepsilon$$

$$\Rightarrow \quad \int_a^b f - \varepsilon < S(f;\dot{\mathcal{P}}) \quad \& \quad S(g;\dot{\mathcal{P}}) < \int_a^b g + \varepsilon.$$

Now $f(x) \leq g(x)$, $\forall x \in [a,b] \Rightarrow$

$$S(f;\dot{\mathcal{P}}) = \sum_{i=1}^n f(t_i)(x_i - x_{i-1}) \leq \sum_{i=1}^n g(t_i)(x_i - x_{i-1}) = S(g;\dot{\mathcal{P}})$$

$$\therefore \int_a^b f - \varepsilon < S(f;\dot{\mathcal{P}}) \leq S(g;\dot{\mathcal{P}}) < \int_a^b g + \varepsilon$$

or $\displaystyle\int_a^b f < \int_a^b g + 2\varepsilon.$

Since $\varepsilon > 0$ is arbitrary, $\displaystyle\int_a^b f \leq \int_a^b g$ ※

Boundedness Theorem

$\boxed{\underline{\text{Thm 7.1.6}} \quad f \in \mathcal{R}[a,b] \Rightarrow f \text{ is bounded on } [a,b].}$

(of course, $f \in \mathcal{R}[a,b] \not\Leftarrow f$ is bounded on $[a,b]$, see later section)

Pf Let $f \in \mathcal{R}[a,b]$ and $\int_a^b f = L$.

And suppose on the contrary that f is unbounded on $[a,b]$.

$f \in \mathcal{R}[a,b]$ with $\int_a^b f = L$ (Take $\varepsilon = 1$ in the def)

$\Rightarrow \exists \delta > 0$ such that

$\forall \dot{\mathcal{P}}$ with $\|\dot{\mathcal{P}}\| < \delta$,

$$|S(f; \dot{\mathcal{P}}) - L| < 1$$

$$\Rightarrow |S(f; \dot{\mathcal{P}})| < |L| + 1 \qquad (*)_1$$

If $\mathcal{P} = \{[x_{i-1}, x_i]\}_{i=1}^n$ be a partition of $[a,b]$.

Then f unbounded

$\Rightarrow \exists$ a subinterval $[x_{i_0-1}, x_{i_0}]$ s.t.

f is unbounded on $[x_{i_0-1}, x_{i_0}]$.

Therefore, we can find t_{i_0} such that

$$|f(t_{i_0})(x_{i_0} - x_{i_0-1})| > |L| + 1 + |\sum_{i \neq i_0} f(x_i)(x_i - x_{i-1})| \quad (*)_2$$

Then the corresponding tagged partition $\dot{\mathcal{P}} = \{[x_{i-1}, x_i]; t_i\}_{i=1}^{n}$,

with tags $\begin{cases} t_i = x_i & \text{for } i \neq i_0 \\ t_{i_0} \end{cases}$

gives Riemann sum

$$S(f;\dot{\mathcal{P}}) = f(t_{i_0})(x_{i_0} - x_{i_0-1}) + \sum_{i \neq i_0} f(x_i)(x_i - x_{i-1})$$

$$\Rightarrow \quad f(t_{i_0})(x_{i_0} - x_{i_0-1}) = S(f;\dot{\mathcal{P}}) - \sum_{i \neq i_0} f(x_i)(x_i - x_{i-1})$$

$$\Rightarrow \quad |f(t_{i_0})(x_{i_0} - x_{i_0-1})| \le |S(f;\dot{\mathcal{P}})| + \left|\sum_{i \neq i_0} f(x_i)(x_i - x_{i-1})\right|$$

$$(\text{by } (*)_1) \quad < |L| + 1 + \left|\sum_{i \neq i_0} f(x_i)(x_i - x_{i-1})\right|$$

which contradicts $(*)_2$

$\therefore$ f must be bounded. ※

<u>eg 7.1.7</u> Thomae's function

$$h(x) = \begin{cases} \frac{1}{n}, & \text{if } x = \frac{m}{n} \in [0,1] \text{ \& } m, n \text{ have no common factors } (m \neq 0,\ n \in \mathbb{N} = \{1,2,3,\cdots\}) \\ & (\gcd(m,n) = 1) \\ 1, & \text{if } x = 0 \\ 0, & \text{if } x \text{ is irrational \& } x \in [0,1]. \end{cases}$$

Then $h \in \mathcal{R}[0,1]$ & $\int_a^b h = 0$

<u>Note</u>: h is discontinuous at every rational number in $[0,1]$

& continuous at every irrational number in $[0,1]$.

(see eg 5.1.6(h) of the Textbook)

<u>Pf</u>: (Similar to eg 7.1.4(d))

$\forall \varepsilon > 0$, the set $E_\varepsilon = \{x \in [0,1] : h(x) \geq \frac{\varepsilon}{2}\}$ is a finite set

(For instance $\frac{\varepsilon}{2} = \frac{1}{5}$, then $E_\varepsilon = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\}$)

Let $N_\varepsilon = \#$ of elements in E_ε.

Define $\delta_\varepsilon = \frac{\varepsilon}{4N_\varepsilon} > 0$.

Then $\forall \dot{\mathcal{P}} = \{[x_{i-1}, x_i]; t_i\}_{i=1}^n$ with $\|\dot{\mathcal{P}}\| < \delta_\varepsilon$,

$$S(h;\dot{\mathcal{P}}) = \sum_{\substack{i=1 \\ t_i \notin E_\varepsilon}}^{n} h(t_i)(x_i - x_{i-1}) + \sum_{\substack{i=1 \\ t_i \in E_\varepsilon}}^{n} h(t_i)(x_i - x_{i-1})$$

($t_i \in$ at most 2 subinterval $\Rightarrow$ at most $2N_\varepsilon$ terms)

$$< \sum_{i=1}^{n} \frac{\varepsilon}{2}(x_i - x_{i-1}) + 2N_\varepsilon \|\dot{\mathcal{P}}\| \qquad (h(x) \leq 1)$$

$$< \frac{\varepsilon}{2} + 2N_\varepsilon \frac{\varepsilon}{4N_\varepsilon} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since clearly $S(h;\dot{\mathcal{P}}) \geq 0$ & $\varepsilon > 0$ is arbitrary,

we have $h \in \mathcal{R}[a,b]$ & $\int_a^b h = 0$.

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