Ch 7 The Riemann Integral

§7.1 Riemann Integral

Def: If $I = [a,b]$ is a closed interval, then a partition of I is a finite, ordered set

$$\mathcal{P} = (x_0, x_1, \cdots, x_n)$$

of points in I such that

$$a = x_0 < x_1 < \cdots < x_n = b$$

Note: A partition $\mathcal{P} = (x_0, x_1, \cdots, x_n)$ is used to divide I into (interior) non-overlapping subintervals:

$$I_1 = [x_0, x_1], \ I_2 = [x_1, x_2], \cdots, I_n = [x_{n-1}, x_n]$$

Hence an alternate notation for $\mathcal{P}$ is

$$\boxed{\mathcal{P} = \{[x_{i-1}, x_i]\}_{i=1}^{n}}$$

Def: The norm (or mesh) of $\mathcal{P} = \{[x_{i-1}, x_i]\}_{i=1}^{n}$ is defined by

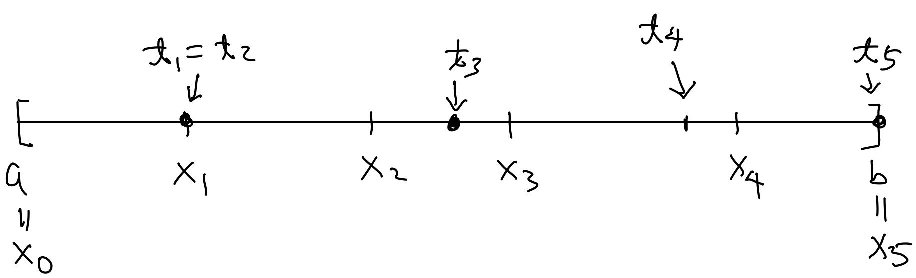
$$\begin{aligned} \|\mathcal{P}\| &= \max_{i=1,\cdots,n} \{x_i - x_{i-1}\} \\ &= \max\{x_1 - x_0, x_2 - x_1, \cdots, x_n - x_{n-1}\} \end{aligned}$$

(= length of largest subinterval)

Def: (1) If $t_i \in I_i = [x_{i-1}, x_i]$, $\forall i = 1, \dots, n$ has been selected of each subinterval of a partition $\mathcal{P} = \{[x_{i-1}, x_i]\}_{i=1}^{n}$ of $I = [a,b]$, then t_i are called *tags* of I_i.

(2) The partition $\mathcal{P} = \{[x_{i-1}, x_i]\}_{i=1}^{n}$, together with tags t_i is called a *tagged partition* of $I = [a,b]$ and is denoted by

$$\dot{\mathcal{P}} = \{ [x_{i-1}, x_i], t_i \}_{i=1}^{n}$$

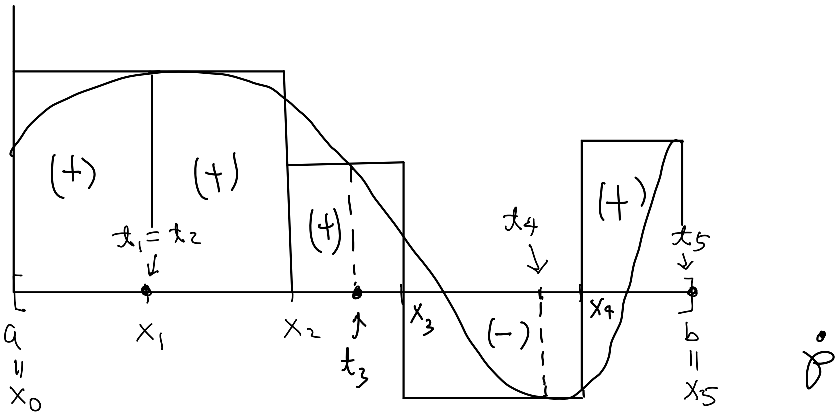


Def: If $\dot{\mathcal{P}} = \{ [x_{i-1}, x_i], t_i \}_{i=1}^{n}$ is a *tagged partition* of $I = [a,b]$, then the *Riemann sum* of a function $f : [a,b] \to \mathbb{R}$ is defined by

(may not "continuous")

$$S(f; \dot{\mathcal{P}}) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$$

Remark: This definition works for the case that $\dot{\mathcal{P}}$ is a *subset* of a partition, and not the entire partition.



$S(f;\dot{\mathcal{P}})$ = sum of (signed) areas of the n rectangles with bases $[x_{i-1}, x_i]$ & heights t_i, $i = 1, \cdots, n$.

Def 7.1.1 (1) A function $f:[a,b]\to\mathbb{R}$ is said to be Riemann integrable on $[a,b]$ if

$\exists\, L\in\mathbb{R}$ such that

$\forall\, \varepsilon>0$, $\exists\, \delta_\varepsilon>0$ such that

$\forall$ tagged partition $\dot{\mathcal{P}}$ of $[a,b]$ with $\|\dot{\mathcal{P}}\|<\delta_\varepsilon$,

$$|S(f;\dot{\mathcal{P}})-L|<\varepsilon.$$

(2) The set of all Riemann integrable functions on $[a,b]$ will be denoted by $\mathcal{R}[a,b]$.

(3) If $f\in\mathcal{R}[a,b]$, the number L is uniquely determined (Thm 7.1.2), called the Riemann integral of f over $[a,b]$, & is denoted by

$$\int_a^b f \quad \text{or} \quad \int_a^b f(x)\,dx$$

(x is a dummy variable, can be replaced by any other notation)

Remark: One often says that L is "the limit" of $S(f;\dot{\mathcal{P}})$ as $\|\dot{\mathcal{P}}\| \to 0$. However $S(f;\dot{\mathcal{P}})$ is not a function of $\|\dot{\mathcal{P}}\|$, it is not the limit (of functions) that defined before. (there are many $\dot{\mathcal{P}}$ with same $\|\dot{\mathcal{P}}\|$)

Thm 7.1.2 If $f \in \mathcal{R}[a,b]$, then the value of the integral is uniquely determined.

Pf: Suppose L' and L'' both satisfy the definition 7.1.1.

Then $\forall \varepsilon > 0$, $\exists\ \delta'_{\frac{\varepsilon}{2}} > 0$ such that

$$|S(f;\dot{\mathcal{P}}_1) - L'| < \frac{\varepsilon}{2} \quad \forall\ \dot{\mathcal{P}}_1 \text{ with } \|\dot{\mathcal{P}}_1\| < \delta'_{\frac{\varepsilon}{2}}$$

and $\exists\ \delta''_{\frac{\varepsilon}{2}} > 0$ such that

$$|S(f;\dot{\mathcal{P}}_2) - L''| < \frac{\varepsilon}{2} \quad \forall\ \dot{\mathcal{P}}_2 \text{ with } \|\dot{\mathcal{P}}_2\| < \delta''_{\frac{\varepsilon}{2}}.$$

Let $\delta_\varepsilon = \min\{\delta'_{\frac{\varepsilon}{2}}, \delta''_{\frac{\varepsilon}{2}}\} > 0$.

If $\dot{\mathcal{P}}$ is a tagged partition with $\|\dot{\mathcal{P}}\| < \delta_\varepsilon$,

then $\|\dot{\mathcal{P}}\| < \delta'_{\varepsilon/2}$ and $\|\dot{\mathcal{P}}\| < \delta''_{\varepsilon/2}$.

Hence $|S(f;\dot{\mathcal{P}}) - L'| < \frac{\varepsilon}{2}$ and $|S(f;\dot{\mathcal{P}}) - L''| < \frac{\varepsilon}{2}$.

$$\Rightarrow \ |L' - L''| \le |S(f;\dot{\mathcal{P}}) - L'| + |S(f;\dot{\mathcal{P}}) - L''|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have $L' = L''$. #

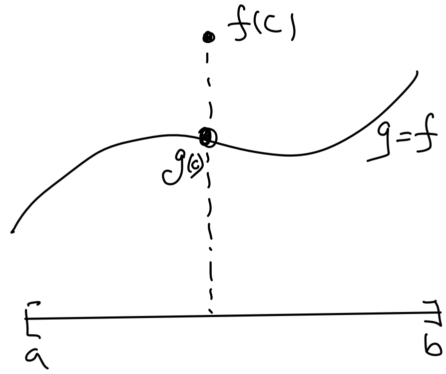
<u>Thm 7.1.3</u> If
- $g \in \mathcal{R}[a,b]$ (Riemann integrable)
- $f(x) = g(x)$ except for a <u>finite number of points</u>.

Then
- $f \in \mathcal{R}[a,b]$ and
- $\int_a^b f = \int_a^b g$.

<u>Pf</u>: Only need to prove the case that

$f(x) = g(x)$ except for <u>one point</u> in $[a,b]$.

Then induction implies the Theorem.



Let c be the point in $[a,b]$

s.t. $f(c) \neq g(c)$.

Then $f(x) = g(x)$, $\forall x \in [a,b] \setminus \{c\}$.

Let $L = \int_a^b g$. (By assumption that $g \in \mathcal{R}[a,b]$, it exists)

For any tagged partition $\dot{\mathcal{P}} = \{[x_{i-1}, x_i], t_i\}_{i=1}^n$,

then (i) $c \in (x_{i_0-1}, x_{i_0})$ for some $i_0 \in \{1, 2, \cdots, n\}$

or (ii) $c = x_{i_0}$ for some $i_0 \in \{1, 2, \cdots, n\}$.

For case (i), $\quad f(x) = g(x)$ for all $[x_{i-1}, x_i]$, $i \neq i_0$

$$\Rightarrow \quad f(t_i) = g(t_i)$$

And hence

$$S(f;\dot{P}) - S(g;\dot{P}) = \sum_{i \neq i_0} f(t_i)(x_i - x_{i-1}) - \sum_{i \neq i_0} g(t_i)(x_i - x_{i-1})$$
$$+ f(t_{i_0})(x_{i_0} - x_{i_0-1}) - g(t_{i_0})(x_{i_0} - x_{i_0-1})$$
$$= \left(f(t_{i_0}) - g(t_{i_0})\right)(x_{i_0} - x_{i_0-1})$$

$$\Rightarrow \quad |S(f;\dot{P}) - S(g;\dot{P})| \leq |f(t_{i_0}) - g(t_{i_0})|\,|x_{i_0} - x_{i_0-1}|$$
$$\leq \left(|f(c)| + |g(c)|\right) \|\dot{P}\| .$$

Similarly for case (ii)

$$S(f;\dot{P}) - S(g;\dot{P}) = \sum_{\substack{i \neq i_0 \\ i_0+1}} \left[f(t_i) - g(t_i)\right](x_i - x_{i-1})$$
$$+ f(t_{i_0})(x_{i_0} - x_{i_0-1}) - g(t_{i_0})(x_{i_0} - x_{i_0-1})$$
$$+ f(t_{i_0+1})(x_{i_0+1} - x_{i_0}) - g(t_{i_0-1})(x_{i_0+1} - x_{i_0})$$
$$= \left(f(t_{i_0}) - g(t_{i_0})\right)(x_{i_0} - x_{i_0-1}) + \left(f(t_{i_0+1}) - g(t_{i_0+1})\right)(x_{i_0+1} - x_{i_0})$$

$$\therefore \quad |S(f;\dot{P}) - S(g;\dot{P})| \leq \left(|f(c)| + |g(c)|\right)\|\dot{P}\| + \left(|f(c)| + |g(c)|\right)\|\dot{P}\|$$
$$= 2\left(|f(c)| + |g(c)|\right)\|\dot{P}\| .$$

Hence, in both cases,

$$|S(f;\dot{P}) - S(g;\dot{P})| \leq 2\left(|f(c)| + |g(c)|\right)\|\dot{P}\|$$

Therefore, $\forall \varepsilon > 0$, for $\delta_1 = \dfrac{\varepsilon}{5(|f(c)|+|g(c)|)}$, we have

$\forall \dot{\mathcal{P}}$ with $\|\dot{\mathcal{P}}\| < \delta_1$,

$$|S(f;\dot{\mathcal{P}}) - S(g;\dot{\mathcal{P}})| \leq 2(|f(c)|+|g(c)|) \cdot \frac{\varepsilon}{5(|f(c)|+|g(c)|)}$$
$$< \frac{\varepsilon}{2}.$$

Now, by $g \in \mathcal{R}[a,b]$ & $L = \int_a^b g$, $\exists\, \delta_2 > 0$ s.t.

$\forall \dot{\mathcal{P}}$ with $\|\dot{\mathcal{P}}\| < \delta_2$,

$$|S(g;\dot{\mathcal{P}}) - L| < \frac{\varepsilon}{2}.$$

Letting $\delta = \min\{\delta_1, \delta_2\} > 0$, we have

$\forall \dot{\mathcal{P}}$ with $\|\dot{\mathcal{P}}\| < \delta$,

$$|S(f;\dot{\mathcal{P}}) - L| \leq |S(f;\dot{\mathcal{P}}) - S(g;\dot{\mathcal{P}})| + |S(g;\dot{\mathcal{P}}) - L|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\therefore$ $f \in \mathcal{R}[a,b]$ and $\int_a^b f = L = \int_a^b g$ ※

Eg 7.1.4

(a) If $f \equiv$ const., then $f \in \mathcal{R}[a,b]$

Pf: Let the const. be k.

Then $f(x) = k \quad \forall x \in [a,b]$

If $\dot{\mathcal{P}} = \{[x_{i-1}, x_i], t_i\}_{i=1}^{n}$ be a tagged partition of $[a,b]$,

then corresponding Riemann sum

$$S(f; \dot{\mathcal{P}}) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$$

$$= \sum_{i=1}^{n} k(x_i - x_{i-1})$$

$$= k(b-a)$$

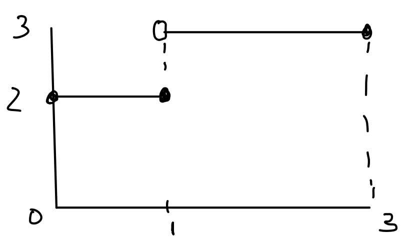
$\therefore \forall \varepsilon > 0$, we can just pick any $\delta > 0$ and have

$$|S(f; \dot{\mathcal{P}}) - k(b-a)| = 0 < \varepsilon, \quad \forall \dot{\mathcal{P}} \text{ with } \|\dot{\mathcal{P}}\| < \delta$$

$\therefore \quad f \equiv k \in \mathcal{R}[a,b]$.

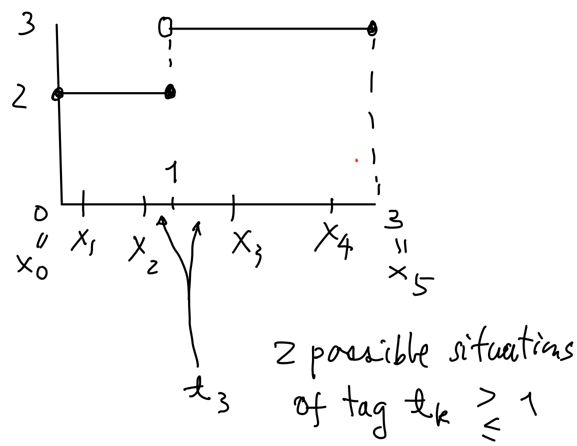
In fact, we have proved that $\int_a^b k = k(b-a)$ ※

(b) $g: [0,3] \to \mathbb{R}$ defined by $g(x) = \begin{cases} 3, & 1 < x \leq 3 \\ 2, & 0 \leq x \leq 1 \end{cases}$



is (Riemann) integrable & $\int_0^3 g = 8$.

Pf: Let $\dot{\mathcal{P}} = \{[x_{i-1}, x_i], t_i\}_{i=1}^{n}$



Let $k = 1, \dots, n$ such that

$0 \le t_1 \le \cdots \le t_k \le 1$ and

$1 < t_{k+1} \le \cdots \le t_n \le 3$

Let $\dot{\mathcal{P}}_1 = \{[x_{i-1}, x_i], t_i\}_{i=1}^{k}$

and $\dot{\mathcal{P}}_2 = \{[x_{i-1}, x_i], t_i\}_{i=k+1}^{n}$

(Using the remark of the definition of Riemann Sum)

we have $$S(g;\dot{\mathcal{P}}) = \sum_{i=1}^{k} g(t_i)(x_i - x_{i-1}) + \sum_{i=k+1}^{n} g(t_i)(x_i - x_{i-1})$$

$$= S(g;\dot{\mathcal{P}}_1) + S(g;\dot{\mathcal{P}}_2)$$

Suppose that $\|\dot{\mathcal{P}}\| < \delta$ for some $\delta > 0$.

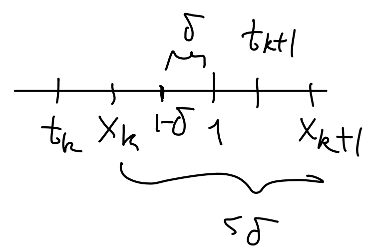
Then $t_k \le 1$, $x_{k-1} \le t_k \le x_k$ and $x_k - x_{k-1} < \delta$,

we have $$x_k < \delta + x_{k-1} \le \delta + t_k \le 1 + \delta$$

$\therefore$ $$U_1 = \bigcup_{i=1}^{k} [x_{i-1}, x_i] = [0, x_k] \subset [0, 1+\delta].$$

(notation in Textbook)

On the other hand, consider $1 - \delta > x_k$.



From the choice of k, $t_{k+1} > 1$.

$\therefore$ $x_{k+1} \ge t_{k+1} > 1$

Hence $\delta > x_{k+1} - x_k \ge 1 - (1-\delta) = \delta$, which is a contradiction.

$\therefore$ $1 - \delta \le x_k$.

Together we have

$$[0, 1-\delta] \subset U_1 = \bigcup_{i=1}^{k} [x_{i-1}, x_i] = [0, x_k] \subset [0, 1+\delta]$$

Therefore

$$S(g; \dot{\mathcal{P}}_1) = \sum_{i=1}^{k} g(t_i)(x_i - x_{i-1})$$

$$= 2x_k \qquad (t_i \leq 1 \Rightarrow g(t_i) = 2)$$

$$\Rightarrow \qquad 2(1-\delta) \leq S(g; \dot{\mathcal{P}}_1) \leq 2(1+\delta) \text{ —— } (*)_1.$$

Similarly,

$$S(g; \dot{\mathcal{P}}_2) = \sum_{i=k+1}^{n} g(t_i)(x_i - x_{i-1})$$

$$= 3(3 - x_k) \qquad (t_i > 1 \Rightarrow g(t_i) = 3)$$

$$\Rightarrow \quad 3(3-(1+\delta)) \leq S(g; \dot{\mathcal{P}}_2) \leq 3(3-(1-\delta))$$

$$3(2-\delta) \leq S(g; \dot{\mathcal{P}}_2) \leq 3(2+\delta) \text{ —— } (*)_2$$

By $(*)_1 + (*)_2$, we have, for $\dot{\mathcal{P}}$ satisfying $\|\dot{\mathcal{P}}\| < \delta$,

$$2(1-\delta) + 3(2-\delta) \leq S(g; \dot{\mathcal{P}}) \leq 2(1+\delta) + 3(2+\delta)$$

i.e. $$8 - 5\delta \leq S(g; \dot{\mathcal{P}}) \leq 8 + 5\delta$$

$$\therefore \qquad |S(g; \dot{\mathcal{P}}) - 8| \leq 5\delta.$$

Therefore $\forall \varepsilon > 0$, we can take $\delta_\varepsilon = \frac{\varepsilon}{10} > 0$ to have $\forall \dot{\mathcal{P}}$ with $\|\dot{\mathcal{P}}\| < \delta_\varepsilon$, $|S(g; \dot{\mathcal{P}}) - 8| \leq 5 \cdot \frac{\varepsilon}{10} < \varepsilon$. ※

(c) $h(x) = x$ (for $x \in [0,1]$) $\in \mathcal{R}[0,1]$ & $\int_0^1 h = \frac{1}{2}$.

Pf: Let $\mathcal{P} = \{[x_{i-1}, x_i]\}_{i=1}^n$ be a partition of I.

Take tags $t_i = q_i$ be the mid-points,

i.e. $$q_i = \frac{x_{i-1} + x_i}{2}.$$

Then the corresponding tagged partition $\dot{Q} = \{[x_{i-1}, x_i]; q_i\}_{i=1}^n$ has Riemann sum

$$S(h;\dot{Q}) = \sum_{i=1}^n h(q_i)(x_i - x_{i-1}) = \sum_{i=1}^n q_i(x_i - x_{i-1})$$

$$= \sum_{i=1}^n \frac{1}{2}(x_i + x_{i-1})(x_i - x_{i-1}) = \frac{1}{2}\sum_{i=1}^n (x_i^2 - x_{i-1}^2)$$

$$= \frac{1}{2}\left[(x_1^2 - x_0^2) + (x_2^2 - x_1^2) + \cdots + (x_n^2 - x_{n-1}^2)\right]$$

$$= \frac{1}{2}(x_n^2 - x_0^2) = \frac{1}{2} \qquad (x_n = 1,\ x_0 = 0)$$

Now if $\dot{\mathcal{P}} = \{[x_{i-1}, x_i], t_i\}_{i=1}^n$ is a tagged partition with the same partition but arbitrary tags t_i, then $\|\dot{\mathcal{P}}\| = \|\dot{Q}\| < \delta$, and

$$|S(h;\dot{\mathcal{P}}) - S(h,\dot{Q})| = \left|\sum_{i=1}^n h(t_i)(x_i - x_{i-1}) - \sum_{i=1}^n h(q_i)(x_i - x_{i-1})\right|$$

$$= \left|\sum_{i=1}^n (t_i - q_i)(x_i - x_{i-1})\right|$$

$$\le \sum_{i=1}^{n} |t_i - q_i| (x_i - x_{i-1})$$

$$< \delta \sum_{i=1}^{n} (x_i - x_{i-1})$$

$$= \delta \qquad \text{since } t_i, q_i \in [x_{i-1}, x_i] \text{ and } x_i - x_{i-1} < \delta$$

Using $S(h; \dot{Q}) = \frac{1}{2}$ for any partition with mid-pt tags, we have $\forall\, \dot{\mathcal{P}}$ with $\|\dot{\mathcal{P}}\| < \delta$,

$$|S(h; \dot{\mathcal{P}}) - \tfrac{1}{2}| < \delta .$$

Hence $\forall\, \varepsilon > 0$, take $\delta_\varepsilon = \varepsilon > 0$, we have

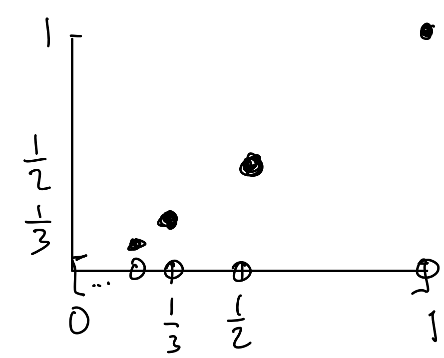
$\dot{\mathcal{P}}$ with $\|\dot{\mathcal{P}}\| < \delta_\varepsilon \Rightarrow |S(h; \dot{\mathcal{P}}) - \frac{1}{2}| < \varepsilon$

$\therefore\ h \in \mathcal{R}[0,1]$ & $\int_a^b h = \frac{1}{2}$. ⨳

(d) $G(x) = \begin{cases} \frac{1}{n}, & \text{if } x = \frac{1}{n} \ (n = 1, 2, \cdots) \\ 0, & \text{elsewhere on } [0,1] \end{cases}$

is (Riemann) integrable on $[0,1]$

and $\int_0^1 G = 0$.



Pf: $\forall\, \varepsilon > 0$, $E_\varepsilon = \{ x \in [0,1] : G(x) \ge \varepsilon \}$

$= \{ 1, \frac{1}{2}, \cdots, \frac{1}{N_\varepsilon} \}$ where $N_\varepsilon = [\frac{1}{\varepsilon}]$ the largest integer $\le \frac{1}{\varepsilon}$.

is a finite set.

Let $\delta = \frac{\varepsilon}{2N_\varepsilon} > 0$.

If $\dot{\mathcal{P}} = \{[x_{i-1}, x_i], t_i\}_{i=1}^{n}$ is a tagged partition with $\|\dot{\mathcal{P}}\| < \delta$.

Then $$S(G; \dot{\mathcal{P}}) = \sum_{i=1}^{n} G(t_i)(x_i - x_{i-1})$$

$$= \sum_{\substack{i=1 \\ t_i \notin E_\varepsilon}}^{n} G(t_i)(x_i - x_{i-1}) + \sum_{\substack{i=1 \\ t_i \in E_\varepsilon}}^{n} G(t_i)(x_i - x_{i-1})$$

$t_i \notin E_\varepsilon \Rightarrow \quad 0 \le G(t_i) < \varepsilon$

$$\therefore \quad 0 \le \sum_{\substack{i=1 \\ t_i \notin E_\varepsilon}}^{n} G(t_i)(x_i - x_{i-1}) < \varepsilon \sum_{i=1}^{n} (x_i - x_{i-1}) = \varepsilon$$

There are only N_ε number of pts in E_ε, & $0 \le G(x) \le 1$, and a tag belongs to at most two subintervals

$$\therefore \quad 0 \le \sum_{\substack{i=1 \\ t_i \in E_\varepsilon}}^{n} G(t_i)(x_i - x_{i-1}) < \sum_{\substack{i=1 \\ t_i \in E_\varepsilon}}^{n} \delta = 2N_\varepsilon \cdot \delta = \varepsilon$$

at most $2N_\varepsilon$ terms

Hence $0 \le S(G; \dot{\mathcal{P}}) < \varepsilon + \varepsilon = 2\varepsilon$, for any $\dot{\mathcal{P}}$ with $\|\dot{\mathcal{P}}\| < \delta_\varepsilon$.

Since $\varepsilon > 0$ is arbitrary, we have $G \in \mathcal{R}[0,1]$ and $\int_0^1 G = 0$. $\divideontimes$