7.4 Q.9.

Let $P_n^1, P_n^2 \in \mathcal{P}(I)$ s.t. $L(f_1, P_n^1) \to L(f_1)$

$L(f_2, P_n^2) \to L(f_2)$

Let P_n be the common refinement of P_n^1, P_n^2.

Write $P_n = (x_1, \ldots, x_k)$.

$$L(f_1, P_n^1) + L(f_2, P_n^2)$$

$$\leq L(f_1, P_n) + L(f_2, P_n)$$

$$= \sum_{i=1}^{k} \left(\inf_{x \in [x_{i-1}, x_i]} f_1(x) + \inf_{x \in [x_{i-1}, x_i]} f_1(x) \right) (x_i - x_{i-1})$$

$$\leq \sum_{i=1}^{k} \inf_{x \in [x_{i-1}, x_i]} (f_1(x) + f_2(x)) (x_i - x_{i-1})$$

$$= L(f_1 + f_2, P)$$

$$\leq L(f_1 + f_2)$$

Letting $n \to \infty$, we get

$$L(f_1) + L(f_2) \leq L(f_1 + f_2)$$

7.4 Q.10

Let $f_1(x) = \begin{cases} 1, & x \in \mathbb{Q} \cap [0,1] \\ 0, & x \in \mathbb{Q}^c \cap [0,1] \end{cases}$

$f_2(x) = \begin{cases} 1, & x \in \mathbb{Q}^c \cap [0,1] \\ 0, & x \in \mathbb{Q} \cap [0,1] \end{cases}$

Then $f_1(x) + f_2(x) = 1 \quad \forall x \in [0,1]$.

Given any partition $P = (x_1, \dots, x_k)$,

$L(f_1, P) = \sum_{i=1}^{k} \inf_{x \in [x_{i-1}, x_i]} f_1(x)(x_i - x_{i-1}) = 0$ since by density of $\mathbb{Q}^c$,

$\exists\, y_i \in [x_{i-1}, x_i] \cap \mathbb{Q}^c \Rightarrow f_1(y_i) = 0 \Rightarrow \inf_{x \in [x_{i-1}, x_i]} f_1(x) = 0 \quad \forall i$

$L(f_2, P) = \sum_{i=1}^{k} \inf_{x \in [x_{i-1}, x_i]} f_2(x)(x_i - x_{i-1}) = 0$ since by density of $\mathbb{Q}$,

$\exists\, z_i \in [x_{i-1}, x_i] \cap \mathbb{Q} \Rightarrow f(z_i) = 0 \Rightarrow \inf_{x \in [x_i, x_i]} f_2(x) = 0 \quad \forall i$

$\therefore \quad L(f_1) = \sup_{P \in \mathcal{P}(I)} L(f_1, P) = 0$

$L(f_2) = \sup_{P \in \mathcal{P}(I)} L(f_2, P) = 0$

However,

$L(f_1 + f_2, P) = \sum_{i=1}^{k} 1 \cdot (x_i - x_{i-1}) = 1$

$\therefore \quad L(f_1 + f_2) = \sup_{P \in \mathcal{P}(I)} L(f_1 + f_2, P) = 1$

$\therefore \quad L(f_1) + L(f_2) = 0 < 1 = L(f_1 + f_2)$.

7.4 Q.12

Since f is strictly increasing on $[0,1]$,

$$\inf_{x\in[\frac{i-1}{n},\frac{i}{n}]} f(x) = \left(\frac{i-1}{n}\right)^2$$

$$\sup_{x\in[\frac{i-1}{n},\frac{i}{n}]} f(x) = \left(\frac{i}{n}\right)^2$$

for $i=1,\dots,n$.

$$\therefore L(f,P_n) = \sum_{i=1}^{n}\left(\frac{i-1}{n}\right)^2\left(\frac{1}{n}\right) = \frac{1}{n^3}\sum_{i=1}^{n}(i-1)^2$$

$$= \frac{1}{n^3}\sum_{j=1}^{n-1} j^2$$

$$= \frac{1}{n^3}\cdot\frac{1}{6}(n-1)(n)(2n-1)$$

$$= \frac{(n-1)(2n-1)}{6n^2}$$

$$U(f,P_n) = \sum_{i=1}^{n}\left(\frac{i}{n}\right)^2\cdot\frac{1}{n} = \frac{1}{n^3}\sum_{i=1}^{n} i^2$$

$$= \frac{1}{n^3}\cdot\frac{1}{6}n(n+1)(2n+1)$$

$$= \frac{(n+1)(2n+1)}{6n^2}$$

$\forall n$

$$\lim_{n\to\infty} L(f,P_n) = \lim_{n\to\infty}\frac{(n-1)(2n-1)}{6n^2} = \lim_{n\to\infty}\frac{(1-\frac{1}{n})(2-\frac{1}{n})}{6} = \frac{1}{3}$$

$L(f,P_n) \leq L(f) \Rightarrow \frac{1}{3} \leq L(f)$. by letting $n\to\infty$.

$$\lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2} = \lim_{n\to\infty} \frac{(1+\frac{1}{n})(2+\frac{1}{n})}{6} = \frac{1}{3}.$$

$U(f, P_n) \geq U(f) \Rightarrow \frac{1}{3} \geq U(f)$ by letting $n\to\infty$.

$\therefore \frac{1}{3} \geq U(f) \geq L(f) \geq \frac{1}{3}$

$\therefore U(f) = L(f) = \frac{1}{3}$

8.1 Q.11

$$\left|\frac{x}{x+n} - 0\right| = \frac{x}{x+n} \quad \text{since } x \geq 0.$$

Let $\varepsilon > 0$. Note that on $[0, a]$,

$$\frac{x}{x+n} \leq \frac{a}{n}$$

Let $N \in \mathbb{N}$ s.t. $\frac{a}{n} < \varepsilon$ as $n \geq N$

Then $\forall x \in [0, a]$, $\frac{x}{x+n} \leq \frac{a}{n} < \varepsilon$ as $n \geq N$.

$\therefore \left(\frac{x}{x+n}\right)$ converges uniformly on $[0, a]$.

However, note that for $x > 0$, $\frac{x}{x+n} = \frac{1}{1+\frac{n}{x}}$ is increasing.

$$\therefore \left\|\frac{x}{x+n}\right\|_{[0,\infty)} = \lim_{x\to\infty} \frac{x}{x+n} = 1 \quad \forall n.$$

$$\therefore \left\|\frac{x}{x+n}\right\|_{[0,\infty)} \not\to 0 \text{ as } n \to \infty$$

$\therefore \left(\frac{x}{x+n}\right)$ does not converge uniformly on $[0, \infty)$

8.1 Q.12

Let $f_n(x) = \dfrac{nx}{1+n^2x^2}$

$$f_n'(x) = \frac{(1+n^2x^2)(n) - (nx)(2n^2x)}{(1+n^2x^2)^2}$$

$$= \frac{(1+n^2x^2)(n) - (nx)(2n^2x)}{(1+n^2x^2)^2}$$

$$= \frac{n - n^3x^2}{(1+n^2x^2)^2}$$

$\therefore f_n'(x) \begin{cases} >0 & \text{when } 0 < x < \frac{1}{n} \\ <0 & \text{when } x > \frac{1}{n} \end{cases} \Rightarrow f_n$ attains global max. at $x = \frac{1}{n}$ on $[0,\infty)$.

For each $a>0$, let $N \in \mathbb{N}$ s.t. $\frac{1}{n} < a$. for $n \geq N$

Then for $n \geq N_1$, $f_n'(x) < 0 \Rightarrow f_n'$ is strictly decreasing.

$\therefore \|f_n\|_{[a,\infty)} = f_n(a) = \dfrac{na}{1+n^2a^2} \to 0$ as $n \to \infty$.

$\therefore$ The conv. is uniform on $[a,\infty)$.

However, $\|f_n\|_{[0,\infty)} = f_n(\frac{1}{n}) = \dfrac{n \cdot \frac{1}{n}}{1+n^2 \cdot \frac{1}{n^2}} = \dfrac{1}{2} \not\to 0$ as $n \to \infty$

$\therefore$ The conv. is not uniform on $[0,\infty)$.

8.1 Q.B

When $x \neq 0$,

$$\lim\left(\frac{nx}{1+nx}\right) = \lim\left(\frac{x}{\frac{1}{n}+x}\right) = \frac{x}{x} = 1$$

When $x = 0$,

$$\lim\left(\frac{nx}{1+nx}\right) = 0$$

On $[a,\infty)$,

$$\frac{nx}{1+nx} - 1 = \frac{nx-1-nx}{1+nx} = \frac{-1}{1+nx}$$

$$\left\|\frac{nx}{1+nx} - 1\right\|_{[a,\infty)} = \left\|\frac{1}{1+nx}\right\|_{[a,\infty)} = \frac{1}{1+na} \to 0 \text{ as } n\to\infty.$$

$\therefore$ The conv. is uniform on $[a,\infty)$.

Write $f_n(x) = \dfrac{nx}{1+nx}$, $f(x) = \begin{cases} 1, & x>0 \\ 0, & x=0. \end{cases}$

Then $f_n(x) - f(x) = \begin{cases} -\dfrac{1}{1+nx}, & x>0 \\ 0, & x=0. \end{cases}$

For $x>0$

$$|f_n(x) - f(x)| = \frac{1}{1+nx} \to 1 \quad \text{as} \quad x\to 0^+$$

$$\therefore \quad \|f_n(x) - f(x)\|_{[0,\infty)} = 1 \not\to 0 \quad \text{as } n\to\infty.$$

$\therefore$ The conv. is not uniform on $[0,\infty)$.